Soln of Ex. 13. $(x^3+3xy^2)dx+(y^3+3x^2y)dy=0$ —— (1)

Here $M=x^3+3xy^2$, $N=y^3+3x^2y$

$\therefore \frac{\partial M}{\partial y}=6xy$, $\frac{\partial N}{\partial x}=6xy$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$\therefore$ given d. eqn is exact.

Now, $(x^3+3xy^2)dx+(y^3+3x^2y)dy=0$

or $x^3dx+y^3dy+3xy(ydx+xdy)=0$

or $x^3dx+y^3dy+3xy\,d(xy)=0$

$\therefore \int x^3dx+\int y^3dy+3\int xy\,d(xy)=C$

$\Rightarrow \frac{x^4}{4}+\frac{y^4}{4}+3.\frac{(xy)^2}{2}=C$

or $x^4+y^4+6x^2y^2=4C$ which is reqd. soln.

Soln of Ex. 14. $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$ —— (1)

Here $M=x^2-4xy-2y^2$, $N=y^2-4xy-2x^2$

$\therefore \frac{\partial M}{\partial y}=-4x-4y$, $\frac{\partial N}{\partial x}=-4y-4x$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ so the d. eqn is exact.

Let $u=\int\limits_{y\text{-const.}} M\,dx=\int\limits_{y\text{-const}}(x^2-4xy-2y^2)\,dx$

$=\frac{x^3}{3}-4y.\frac{x^2}{2}-2y^2.x+\phi(y)$

$\Rightarrow u=\frac{x^3}{3}-2x^2y-2xy^2+\phi(y)$ —— (2)

$\therefore \frac{\partial u}{\partial y}=-2x^2-4xy+\phi'(y)$

$\Rightarrow y^2-\cancel{4xy}-\cancel{2x^2}=-\cancel{2x^2}-\cancel{4xy}+\phi'(y)$ $\because \frac{\partial u}{\partial y}=N$

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$$\Rightarrow \varphi'(y) = y^2 \quad \therefore \varphi(y) = \frac{y^3}{3}$$

$\therefore$ solution is given by $u = c$

$$\text{or } \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c$$

$$\text{or } x^3 - 6x^2y - 6xy^2 + y^3 = 3c.$$

Soln of Ex. 15.
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$$(x+y)^2 dx + (x^2+2xy-y^2)dy = 0 \quad \text{—— (1)}$$

Here $M = (x+y)^2 = x^2+2xy+y^2$, $N = x^2+2xy-y^2$

$$\therefore \frac{\partial M}{\partial y} = 2x+2y \quad , \quad \frac{\partial N}{\partial x} = 2x+2y$$

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the d.eqn is exact.

Let $u = \int\limits_{y\text{-const.}} M\,dx = \int\limits_{y\text{-const.}} (x^2+2xy+y^2)\,dx$

$$\Rightarrow u = \frac{x^3}{3} + x^2y + xy^2 + \varphi(y) \quad \text{—— (2)}$$

$$\therefore \frac{\partial u}{\partial y} = x^2 + 2xy + \varphi'(y)$$

$$\Rightarrow \cancel{x^2} + \cancel{2xy} - y^2 = \cancel{x^2} + \cancel{2xy} + \varphi'(y) \quad ; \because \frac{\partial u}{\partial y} = N$$

$$\Rightarrow \varphi'(y) = -y^2$$

$$\therefore \varphi(y) = -\frac{y^3}{3}$$

$\therefore$ Solution is given by $u = c$

$$\text{or } \frac{x^3}{3} + x^2y + xy^2 - \frac{y^3}{3} = c$$

Ex. 16. Solve: $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\dfrac{a^2-x^2-y^2}{x^2+y^2}}$

$$\Rightarrow \frac{\frac{1}{2}(2x\,dx + 2y\,dy)}{x^2 \cdot \frac{x\,dy - y\,dx}{x^2}} = \frac{\sqrt{a^2-(x^2+y^2)}}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \frac{d(x^2+y^2)}{2x^2 \cdot d\left(\frac{y}{x}\right)} = \frac{\sqrt{a^2-(x^2+y^2)}\,\sqrt{x^2+y^2}}{x^2+y^2}$$

$$\Rightarrow \frac{d(x^2+y^2)}{2\cancel{x^2}\, d\left(\frac{y}{x}\right)} = \frac{\sqrt{a^2-(x^2+y^2)}\sqrt{x^2+y^2}}{\cancel{x^2}\left(1+\frac{y^2}{x^2}\right)}$$

$$\Rightarrow \frac{1}{2}\frac{du}{dv} = \frac{\sqrt{a^2-u}\sqrt{u}}{1+v^2} \quad \text{where } u = x^2+y^2,\ v = \frac{y}{x}$$

$$\Rightarrow \frac{du}{2\sqrt{u}\sqrt{a^2-u}} = \frac{dv}{1+v^2}$$

$$\therefore \int \frac{dv}{1+v^2} = \int \frac{du}{2\sqrt{u}\sqrt{a^2-u}} + c$$

Let $t = \sqrt{u}$
or $t^2 = u$
$\therefore 2t\,dt = du$

$$\Rightarrow \tan^{-1} v = \int \frac{2t\,dt}{2t\sqrt{a^2-t^2}} + c$$

$$= \int \frac{dt}{\sqrt{a^2-t^2}} + c$$

$$= \sin^{-1}\frac{t}{a} + c$$

$$= \sin^{-1}\frac{\sqrt{u}}{a} + c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} = \sin^{-1}\frac{\sqrt{x^2+y^2}}{a} + c$$

which is required solution.

Ex. 17. Solve : $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$.

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = \frac{1}{x^2}\sqrt{x^2+y^2}\,dx$$

$$\Rightarrow d\left(\frac{y}{x}\right) = \frac{1}{x}\sqrt{1+\frac{y^2}{x^2}}\,dx$$

$$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \frac{1}{x}dx$$

$$\Rightarrow \frac{dt}{\sqrt{1+t^2}} = \frac{1}{x}dx \quad \text{where } t = \frac{y}{x}$$

$$\therefore \int \frac{dt}{\sqrt{1+t^2}} = \int \frac{1}{x}dx + \log c$$

$$\Rightarrow \log\left(t+\sqrt{1+t^2}\right) = \log x + \log c = \log(cx)$$

$$\Rightarrow t + \sqrt{1+t^2} = cx \Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = cx$$

$$\Rightarrow y + \sqrt{x^2+y^2} = cx^2 \quad \text{which is reqd. soln.}$$

Ex. 18. Solve: $y\,dx - x\,dy = (x^2+y^2)\,dy$

$$\Rightarrow \frac{y\,dx - x\,dy}{y^2} = \frac{x^2+y^2}{y^2}\,dy$$

$$\Rightarrow d\left(\frac{x}{y}\right) = \left\{\left(\frac{x}{y}\right)^2 + 1\right\} dy$$

$$\Rightarrow \frac{d\left(\frac{x}{y}\right)}{1+\left(\frac{x}{y}\right)^2} = dy$$

$$\therefore \int \frac{dt}{1+t^2} = \int dy + c \quad \text{where } t = \frac{x}{y}$$

$$\Rightarrow \tan^{-1} t = y + c$$

$$\Rightarrow \tan^{-1}\frac{x}{y} = y + c \quad \text{which is reqd. soln}$$

or $\frac{x}{y} = \tan(y+c)$

or $x = y\tan(y+c)$

Integrating Factor (I.F.):—

If a differential equation is not exact, but it becomes exact when multiplied by some function then this function is called an integrating factor of the d. eqn.

e.g., $y\,dx - x\,dy = 0$ —— (1) is not an exact d. eqn.

Here $M = y$, $N = -x$

$\therefore \frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$ so that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Multiplying the d. eqn (1) by $\frac{1}{y^2}$ we get,

$$\frac{1}{y}\,dx - \frac{x}{y^2}\,dy = 0 \text{ —— (2)}$$

For this d. eqn (2), $M = \frac{1}{y}$, $N = -\frac{x}{y^2}$

$\therefore \frac{\partial M}{\partial y} = -\frac{1}{y^2}$, $\frac{\partial N}{\partial x} = -\frac{1}{y^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so that the d. eqn (2) is exact.

i.e., the d. eqn (1), which is not exact, now becomes exact after multiplication by the factor $\frac{1}{y^2}$ and hence $\frac{1}{y^2}$ is an I. F. of the d. eqn (1).

Rules for finding I.F. :-

Rule I. :- If $Mdx+Ndy=0$ is not exact i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

and $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = f(x)$, a function of x-alone, then

$$I.F. = e^{\int f(x)dx}.$$

Proof:- Let the d.eqn $Mdx+Ndy=0$ — (1), is not exact.

and let $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = f(x)$, a function of x-alone.

To prove $e^{\int f(x)dx}$ is an I.F. of the d.eqn. (1).

Multiplying the d.eqn (1) by $e^{\int f(x)dx}$ we get —

$$M e^{\int f(x)dx} dx + N e^{\int f(x)dx} dy = 0$$

or $M_1 dx + N_1 dy = 0$ (say) — (2)

Now $\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}\left\{N e^{\int f(x)dx}\right\} = N. e^{\int f(x)dx} \cdot \frac{\partial}{\partial x}\int f(x)dx + e^{\int f(x)dx}\frac{\partial N}{\partial x}$

$= N e^{\int f(x)dx} \cdot f(x) + e^{\int f(x)dx} \cdot \frac{\partial N}{\partial x}$

$= e^{\int f(x)dx}\left[N f(x) + \frac{\partial N}{\partial x}\right]$ $\quad \because \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = f(x)$

$\Rightarrow N f(x) = \frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}.$

$= e^{\int f(x)dx} \cdot \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} + \frac{\partial N}{\partial x}\right]$

$= e^{\int f(x)dx} \frac{\partial M}{\partial y}$

$= \frac{\partial}{\partial y}\left\{M e^{\int f(x)dx}\right\} = \frac{\partial M_1}{\partial y}$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, so the d.eqn (2) is exact & hence $e^{\int f(x)dx}$ is an I.F. of the d.eqn. (1).

Rule. II :- If $Mdx+Ndy=0$ is not exact i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

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and $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = f(y)$, a function of y-alone, then

$$I.F. = e^{\int f(y)dy}.$$

Proof:– Let the d.eqⁿ $Mdx+Ndy=0$ — ①, is not exact.

and let $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=f(y)$, a function of y-alone.

To prove $e^{\int f(y)dy}$ is an I.F. of the d.eqⁿ. ①.

Multiplying the d.eqⁿ ① by $e^{\int f(y)dy}$ we get–

$$Me^{\int f(y)dy}dx+Ne^{\int f(y)dy}dy=0$$

or $$M_1dx+N_1dy=0 \text{ (say)} \text{ — ②}$$

Now $$\frac{\partial M_1}{\partial y}=\frac{\partial}{\partial y}\left\{Me^{\int f(y)dy}\right\}=M.e^{\int f(y)dy}\cdot\frac{\partial}{\partial y}\int f(y)dy+e^{\int f(y)dy}\frac{\partial M}{\partial y}$$

$$=Me^{\int f(y)dy}\cdot f(y)+e^{\int f(y)dy}\cdot\frac{\partial M}{\partial y}$$

$$=e^{\int f(y)dy}\left[Mf(y)+\frac{\partial M}{\partial y}\right], \quad \because \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=f(y)$$

$$=e^{\int f(y)dy}\cdot\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}+\frac{\partial M}{\partial y}\right] \quad \Rightarrow Mf(y)=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}.$$

$$=e^{\int f(y)dy}\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left\{e^{\int f(y)dy}.N\right\}=\frac{\partial N_1}{\partial x}$$

$\therefore \dfrac{\partial M_1}{\partial y}=\dfrac{\partial N_1}{\partial x}$, so the d.eqⁿ ② is exact & hence $e^{\int f(y)dy}$ is an I.F. of the d.eqⁿ. ①.

Rule III:– If the d.eqⁿ $Mdx+Ndy=0$ is not exact i.e., $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ and it is a homogeneous d.eqⁿ i.e., M & N are homogeneous functions of same degree in x & y, and also $Mx+Ny\neq 0$, then the I.F. of this d.eqⁿ is $\dfrac{1}{Mx+Ny}$.

Proof:– Let the d.eqⁿ $Mdx+Ndy=0$ — ①, is not exact i.e., $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ and it is a homogeneous d.eqⁿ i.e., M & N are homogeneous functions of same degree in x & y, say each of degree n.

$\therefore$ $M=x^n\phi\left(\frac{y}{x}\right)$ and $N=x^n\psi\left(\frac{y}{x}\right)$.

$$\therefore \quad \frac{Mx-Ny}{Mx+Ny} = \frac{x^n\{x\phi(\frac{y}{x}) - y\psi(\frac{y}{x})\}}{x^n\{x\phi(\frac{y}{x}) + y\psi(\frac{y}{x})\}} = \frac{\phi(\frac{y}{x}) - \frac{y}{x}\psi(\frac{y}{x})}{\phi(\frac{y}{x}) + \frac{y}{x}\psi(\frac{y}{x})} = f(\tfrac{y}{x}) \text{, say.}$$

Now $\quad Mdx+Ndy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$, Note this step.

$$\therefore \quad \frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}\left[\frac{dx}{x}+\frac{dy}{y}+\frac{Mx-Ny}{Mx+Ny}\cdot\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{xy}(ydx+xdy) - f(\tfrac{y}{x}).\{d(\log y) - d(\log x)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{xy}d(xy) - f(e^{\log\frac{y}{x}})d(\log y - \log x)\right] \quad \because e^{\log\frac{y}{x}} = \frac{y}{x}$$

$$= \frac{1}{2}\left[d\{\log(xy)\} - d\int g(\log\tfrac{y}{x})\,d(\log\tfrac{y}{x})\right]$$

$$= \frac{1}{2}du = d(\tfrac{1}{2}u) \qquad \text{where } f(e^{\log\frac{y}{x}}) = g(\log\tfrac{y}{x}) \text{, say}$$

$\therefore \frac{1}{Mx+Ny}(Mdx+Ndy) = d(\frac{1}{2}u)$ is a perfect differential. where $u = \log(xy) - \int g(\log\frac{y}{x})\,d(\log\frac{y}{x})$

$\therefore$ the d. eq^n $\frac{1}{Mx+Ny}(Mdx+Ndy) = 0$ is exact.

$\therefore \frac{1}{Mx+Ny}$ is an I.F. of the d.eq^n (1).

Rule-IV :– If the d. eq^n $Mdx+Ndy=0$ is not exact, where M is of the form $yf(xy)$ & N is of the form $xg(xy)$, and also $Mx-Ny \neq 0$ then I.F. $= \frac{1}{Mx-Ny}$.

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Proof:– Let $Mdx+Ndy=0$ — (1) is not exact, where

$M = yf(xy)$ & $N = xg(xy)$ and $Mx-Ny \neq 0$.

Now, $\quad Mdx+Ndy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$

$$\therefore \quad \frac{Mdx+Ndy}{Mx-Ny} = \frac{1}{2}\left[\frac{Mx+Ny}{Mx-Ny}\left(\frac{dx}{x}+\frac{dy}{y}\right)+\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{xyf(xy)+xyg(xy)}{xyf(xy)-xyg(xy)}\{d(\log x)+d(\log y)\}+\{d(\log x)-d(\log y)\}\right]$$

$$= \frac{1}{2}\left[\frac{f(xy)+g(xy)}{f(xy)-g(xy)}d(\log x+\log y)+d(\log x-\log y)\right]$$

$= \frac{1}{2}\left[F(xy)\, d\{\log(xy)\} + d\left(\log\frac{x}{y}\right)\right]$ where $\frac{f(xy)+g(xy)}{f(xy)-g(xy)} = F(xy)$, say

$= \frac{1}{2}\left[F\{e^{\log(xy)}\}\, d\{\log(xy)\} + d\left(\log\frac{x}{y}\right)\right]$, $\because e^{\log(xy)} = xy$

$= \frac{1}{2}\left[G\{\log(xy)\}\, d\{\log(xy)\} + d\left(\log\frac{x}{y}\right)\right]$, where $F\{e^{\log(xy)}\} = G\{\log(xy)\}$, say

$= d\left[\frac{1}{2}\int G\{\log(xy)\}\, d\{\log(xy)\} + \frac{1}{2}\log\frac{x}{y}\right]$

$= du$ (say). where $u = \frac{1}{2}\int G\{\log(xy)\}\, d\{\log(xy)\} + \frac{1}{2}\log\frac{x}{y}$

$\therefore \frac{1}{Mx-Ny}(Mdx+Ndy)$ is a perfect differential.

$\therefore$ the d. eqn $\frac{1}{Mx-Ny}(Mdx+Ndy) = 0$ is exact.

$\therefore \frac{1}{Mx-Ny}$ is an I.F. of the d. eqn (1).

Theorem: — The differential equation $Mdx+Ndy=0$,
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which has a solution, possesses an infinite number of integrating factors.

Proof: — Let the d. eqn $Mdx+Ndy=0$ — (1) has a solution $u(x,y)=c$ then either $Mdx+Ndy = du$
or $\mu M dx + \mu N dy = du$ where $\mu = \mu(x,y)$ is a function of x & y and is an I.F. of the d. eqn (1). In particular $\mu(x,y)$ may be equal to 1.

Let $f(u)$ be any function of u,

Then $\mu f(u)(Mdx+Ndy) = f(u)du = d\left(\int f(u)du\right) = dv$, say
where $v = \int f(u)du$.

This shows that d. eqn (1) becomes exact when multiplied by $\mu f(u)$ and therefore $\mu f(u)$ is an I.F.
Since $f(u)$ is arbitrary function of u, it follows that the d. eqn (1) possesses an infinite number of integrating factors.

Ex. 1. Solve: $(2x^2+y^2+x)\,dx + xy\,dy = 0$ —— (1)

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Soln Here $M = 2x^2+y^2+x$, $N = xy$

$\therefore \frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so the d. eq^n (1) is not exact.

Now $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{1}{x}$, a function of x-alone.

$\therefore$ I. F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$ (Rule. I. applied)

Multiplying the d. eq^n (1) by I.F. i.e., by x we get—

$(2x^3 + xy^2 + x^2)\,dx + x^2y\,dy = 0$ —— (2), which is exact.

$\Rightarrow (2x^3+x^2)\,dx + xy(y\,dx + x\,dy) = 0$

$\Rightarrow (2x^3+x^2)\,dx + xy\,d(xy) = 0$, Note that $y\,dx+x\,dy = d(xy)$.

$\therefore \int (2x^3+x^2)\,dx + \int xy\,d(xy) = c$

$\Rightarrow 2\frac{x^4}{4} + \frac{x^3}{3} + \frac{(xy)^2}{2} = c$

$\Rightarrow \frac{x^4}{2} + \frac{x^3}{3} + \frac{1}{2}x^2y^2 = c$ which is general solution of the given d. eq^n.

Ex. 2. Solve: $(y^2e^x + 2xy)\,dx - x^2\,dy = 0$ —— (1)

Soln Here $M = y^2e^x + 2xy$, $N = -x^2$

$\therefore \frac{\partial M}{\partial y} = 2ye^x + 2x$, $\frac{\partial N}{\partial x} = -2x$, so $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ the given d. eq^n is not exact.

Now $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-2x-2ye^x-2x}{y^2e^x+2xy} = \frac{-2(ye^x+2x)}{y(ye^x+2x)} = -\frac{2}{y}$

$\therefore$ I. F. $= e^{\int -\frac{2}{y}dy}$, (by rule II) which is a function of y only.

$= e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$.

Multiplying the d. eq^n (1) by I.F. i.e., by $\frac{1}{y^2}$ we get—

$$\left(e^x + \frac{2x}{y}\right) dx - \frac{x^2}{y^2} dy = 0 \quad \text{—— (2)}, \text{ which is exact.}$$

(because multiplication by I.F. makes d.eqⁿ exact)

$\Rightarrow e^x dx + \frac{1}{y}\cdot 2x dx + x^2 \cdot \left(-\frac{1}{y^2}\right) dy = 0$

$\Rightarrow e^x dx + d\left(x^2 \cdot \frac{1}{y}\right) = 0$, [(Note that $\frac{d}{dx}\left(x^2 \cdot \frac{1}{y}\right) = 2x\cdot\frac{1}{y} + x^2\left(-\frac{1}{y^2}\right)\frac{dy}{dx}$ or $d\left(x^2\cdot\frac{1}{y}\right) = \frac{2x}{y}dx - \frac{x^2}{y^2}dy$.]

$\therefore \int e^x dx + \int d\left(\frac{x^2}{y}\right) = c$

$\Rightarrow e^x + \frac{x^2}{y} = c$ which is required solution.

Ex. 3. Solve : $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$ —— (1)

It is a homogeneous d.eqⁿ of 1st order.

Here $M = x^2y - 2xy^2$, $N = 3x^2y - x^3$ (are both homogeneous fⁿ of degree 3.)

$\therefore \frac{\partial M}{\partial y} = x^2 - 4xy, \quad \frac{\partial N}{\partial x} = 6xy - 3x^2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so the d.eqⁿ (1) is not exact.

Now $Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y = x^2y^2 \neq 0$

$\therefore$ I.F. $= \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$ (by rule - III).

Multiplying the d.eqⁿ (1) by I.F. $= \frac{1}{x^2y^2}$ we get

$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0$ —— (2) which is exact.

$\Rightarrow -\frac{2}{x}dx + \frac{3}{y}dy + \frac{1}{y}dx - \frac{x}{y^2}dy = 0$

$\Rightarrow -\frac{2}{x}dx + \frac{3}{y}dy + d\left(\frac{x}{y}\right) = 0$

$\therefore -2\int \frac{1}{x}dx + 3\int\frac{1}{y}dy + \int d\left(\frac{x}{y}\right) = \log c$

$\Rightarrow -2\log x + 3\log y + \frac{x}{y} = \log c$

$\Rightarrow \log\frac{y^3}{cx^2} = -\frac{x}{y} \Rightarrow \frac{y^3}{cx^2} = e^{-\frac{x}{y}}$

$\Rightarrow y^3 = cx^2 e^{-x/y}$ which is reqd. solⁿ.

Note that — $\frac{1}{y}dx - \frac{x}{y^2}dy$

$= \frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

standard result got earlier.

OR

$\frac{1}{y}dx - \frac{x}{y^2}dy$

$= \frac{1}{y}\cdot dx + x\cdot\left(-\frac{1}{y^2}\right)dy$

$= d\left(x\cdot\frac{1}{y}\right)$

Ex. 4. Find an I.F. of the d. equation
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$(x^2+y^2+x)dx - xy\,dy = 0$ —— (1) & hence solve.

Soln Here $M = x^2+y^2+x$, $N = -xy$

$\therefore \frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = -y$. $\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so the d.eqn is not exact.

Now $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y+y}{-xy} = -\frac{3y}{xy} = -\frac{3}{x}$

$\therefore$ I.F. $= e^{\int -\frac{3}{x}dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3}$.

(by rule I)

Multiplying the given d.eq^n by I.F. $= \frac{1}{x^3}$ we get —

$\left(\frac{1}{x} + \frac{y^2}{x^3} + \frac{1}{x^2}\right)dx - \frac{y}{x^2}dy = 0$ —— (2) which is exact.

$\Rightarrow \left(\frac{1}{x} + \frac{1}{x^2}\right)dx + \frac{y^2}{x^3}dx - \frac{y}{x^2}dy = 0$

$\Rightarrow \left(\frac{1}{x} + \frac{1}{x^2}\right)dx + \frac{y^3}{x^3}\left(\frac{1}{y}dx - \frac{x}{y^2}dy\right) = 0$, (Note this step)

$\Rightarrow \left(\frac{1}{x} + \frac{1}{x^2}\right)dx + \left(\frac{y}{x}\right)^3 \frac{y\,dx - x\,dy}{y^2} = 0$, $\because \frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$.

$\Rightarrow \left(\frac{1}{x} + \frac{1}{x^2}\right)dx + \left(\frac{y}{x}\right)^3 d\left(\frac{x}{y}\right) = 0$

$\therefore \int\left(\frac{1}{x} + \frac{1}{x^2}\right)dx + \int\left(\frac{x}{y}\right)^{-3} d\left(\frac{x}{y}\right) = C$

soln can be written as $\log x - \frac{1}{x} - \frac{y^2}{2x^2} = C$.

$\Rightarrow \log x - \frac{1}{x} - \frac{1}{2}\left(\frac{x}{y}\right)^{-2} = C$ which is reqd. solution.

Ex. 5. Solve: $(1+xy)y\,dx + (1-xy)x\,dy = 0$ —— (1)

Soln. Here $M = (1+xy)y = y(1+xy) = y\,f(xy)$, say

$= y + xy^2$

$\therefore \frac{\partial M}{\partial y} = 1 + 2xy$

$N = (1-xy)x = x(1-xy) = x\,g(xy)$, say

$= x - x^2y$

$\therefore \frac{\partial N}{\partial x} = 1 - 2xy$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ given d. eq^n (1) is not exact.

Now $Mx - Ny = xy + x^2y^2 - (xy - x^2y^2) = 2x^2y^2$

$\therefore$ $I.F. = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$

Multiplying given d.eqn by I.F. we get—

$$\frac{1}{2x^2y^2}(1+xy)y\,dx + \frac{1}{2x^2y^2}(1-xy)x\,dy = 0$$

$$\Rightarrow \left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0 \text{ — (2)}$$ which is exact.

$$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{x^2y^2}(y\,dx + x\,dy) = 0$$

$$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{(xy)^2}d(xy) = 0$$

$\therefore$ $\int \frac{1}{x}dx - \int \frac{1}{y}dy + \int \frac{1}{t^2}dt = C$ where $t = xy$

$$\Rightarrow \log x - \log y - \frac{1}{t} = C$$

$\Rightarrow \log\frac{x}{y} - \frac{1}{xy} = C$ which is reqd. soln.

Ex. 6. Solve: (i) $y\,dx + x(1-3x^2y^2)dy = 0$

(ii) $y(x^2y^2 + xy + 1)dx + x(x^2y^2 - xy + 1)dy = 0$.

Soln (i) Here $M = y$ is of the form $y f(xy)$ where $f(xy) = 1$

$N = x(1-3x^2y^2)$ is of the form $x g(xy)$ where (special case)

Clearly $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ i.e. the $g(xy) = 1 - 3(xy)^2$.

d.eqn (1) is not exact.

Now $Mx - Ny = xy - xy(1-3x^2y^2) = 3x^3y^3 \neq 0$

$\therefore$ $I.F. = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$ (by Rule IV) (Note that the factor $\frac{1}{3}$ may be omitted in practice)

Multiplying the d.eqn (1) by $I.F. = \frac{1}{3x^3y^3}$ we get—

$$\frac{1}{3x^3y^2}dx + \frac{1}{3}\left(\frac{1}{x^2y^3} - \frac{3}{y}\right)dy = 0$$

or $\frac{1}{x^3y^2}dx + \left(\frac{1}{x^2y^3} - \frac{3}{y}\right)dy = 0$ — (2) which is exact.

$$\Rightarrow \frac{1}{x^3y^3}(x\,dy + y\,dx) - \frac{3}{y}\,dy = 0$$

$$\therefore \int (xy)^{-3}\,d(xy) - 3\int \frac{1}{y}\,dy = C$$

$$\Rightarrow -\frac{1}{2x^2y^2} - 3\log y = C \quad \text{or} \quad \frac{1}{2x^2y^2} + 3\log y + C = 0$$

which is reqd. soln.

(ii) Here $M = y(x^2y^2 + xy + 1)$ is of the form $y f(xy)$

$$= x^2y^3 + xy^2 + y$$

$$\therefore \frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1$$

$N = x(x^2y^2 - xy + 1)$ is of the form $x g(xy)$

$$= x^3y^2 - x^2y + x$$

$$\therefore \frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1 \quad \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

$\therefore$ given d. eqn is not exact.

Now $Mx - Ny = x^3y^3 + x^2y^2 + xy - (x^3y^3 - x^2y^2 + xy)$

$$= 2x^2y^2$$

$$\therefore \text{I.F.} = \frac{1}{2x^2y^2}.$$

Multiplying given d.eqn by I.F. we get

$$\frac{1}{2x^2y^2}y(x^2y^2 + xy + 1)\,dx + \frac{1}{2x^2y^2}x(x^2y^2 - xy + 1)\,dy = 0$$

$$\Rightarrow \left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \left(x - \frac{1}{y} + \frac{1}{xy^2}\right)dy = 0 \quad \text{--- (2)}$$

which is exact.

$$\Rightarrow \frac{1}{x}\,dx - \frac{1}{y}\,dy + (y\,dx + x\,dy) + \frac{1}{x^2y^2}(y\,dx + x\,dy) = 0$$

$$\Rightarrow \frac{1}{x}\,dx - \frac{1}{y}\,dy + d(xy) + \frac{1}{(xy)^2}\,d(xy) = 0$$

$$\therefore \int \frac{1}{x}\,dx - \int \frac{1}{y}\,dy + \int d(xy) + \int \frac{1}{(xy)^2}\,d(xy) = C$$

$$\Rightarrow \log x - \log y + xy - \frac{1}{xy} = C \quad \text{which is reqd. soln.}$$

Ex.7. 2022P Write the value of $d[\log_e(xy)]$.

Soln. Let $u = \log_e(xy)$

$$\therefore du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{1}{xy}\cdot y\,dx + \frac{1}{xy}\cdot x\,dy = \frac{1}{x}dx + \frac{1}{y}dy$$

$$\therefore d[\log_e(xy)] = \frac{1}{x}dx + \frac{1}{y}dy.$$

Alt

$$\frac{du}{dx} = \frac{d}{dx}\log_e(xy) = \frac{1}{xy}\frac{d}{dx}(xy) = \frac{1}{xy}\left(x\frac{dy}{dx} + 1\cdot y\right)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{y}\frac{dy}{dx} + \frac{1}{x}$$

$$\therefore du = \frac{1}{y}dy + \frac{1}{x}dx$$

$$\therefore d[\log_e(xy)] = \frac{1}{x}dx + \frac{1}{y}dy.$$

Ex.8. 2022P Solve: $y\,dx - x\,dy + (1+x^2)dx + x^2\sin y\,dy = 0$. (4m)

Soln.

$$y\,dx - x\,dy + (1+x^2)dx + x^2\sin y\,dy = 0$$

$$\Rightarrow \frac{y\,dx - x\,dy}{x^2} + \frac{1+x^2}{x^2}dx + \sin y\,dy = 0$$

$$\Rightarrow -\left(\frac{x\,dy - y\,dx}{x^2}\right) + \left(\frac{1}{x^2}+1\right)dx + \sin y\,dy = 0$$

$$\Rightarrow -d\left(\frac{y}{x}\right) + \left(\frac{1}{x^2}+1\right)dx + \sin y\,dy = 0$$

$$\therefore -\int d\left(\frac{y}{x}\right) + \int (x^{-2}+1)\,dx + \int \sin y\,dy = c$$

$$\Rightarrow -\frac{y}{x} + \frac{x^{-2+1}}{-2+1} + x - \cos y = c$$

$$\Rightarrow -\frac{y}{x} - \frac{1}{x} + x - \cos y = c$$

$$\Rightarrow -y - 1 + x^2 - x\cos y = cx$$

$$\Rightarrow y + 1 - x^2 + x\cos y + cx = 0$$

which is reqd. solution.

Ex.9. 2019P Solve: $(x^3+y^3)dx - xy^2dy = 0$

Soln. $(x^3+y^3)dx - xy^2dy = 0$ —— (1)

Here $M = x^3+y^3$, $N = -xy^2$ are homogeneous functions of degree three each.

$\therefore \frac{\partial M}{\partial y} = 3y^2$, $\frac{\partial N}{\partial x} = -y^2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so the d. eq^n (1) is not exact.

Now, $Mx + Ny = (x^3+y^3)x + (-xy^2)y = x^4 + xy^3 - xy^3 = x^4$

$\therefore$ I.F. $= \frac{1}{Mx+Ny} = \frac{1}{x^4}$.

Multiplying d. eq^n (1) by $\frac{1}{x^4}$ i.e., by I.F. we get

$$\frac{1}{x^4}(x^3+y^3)dx - \frac{1}{x^4}\cdot xy^2 dy = 0$$

$$\Rightarrow \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}dy = 0 \quad \text{—— (2) which is exact.}$$

$$\Rightarrow \frac{1}{x}dx + \frac{y^2}{x^2}\left(\frac{y}{x^2}dx - \frac{1}{x}dy\right) = 0$$

$$\Rightarrow \frac{1}{x}dx + \left(\frac{y}{x}\right)^2 \cdot \frac{ydx - xdy}{x^2} = 0$$

$$\Rightarrow \frac{1}{x}dx + \left(\frac{y}{x}\right)^2 \cdot d\left(\frac{y}{x}\right) = 0$$

$$\therefore \int \frac{1}{x}dx + \int \left(\frac{y}{x}\right)^2 d\left(\frac{y}{x}\right) = c$$

$$\Rightarrow \log x + \frac{1}{3}\cdot\left(\frac{y}{x}\right)^3 = c \quad \text{which is reqd. solution.}$$

Ex. 10. **2022H** Find α and β so that the equation (2m)

$$(\alpha xy^3 + y\cos x)dx + (x^2y^2 + \beta \sin x)dy = 0 \text{ is exact.}$$

Sol^n Here $M = \alpha xy^3 + y\cos x$, $N = x^2y^2 + \beta\sin x$

Given d. eq^n will be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

i.e., if $\frac{\partial}{\partial y}(\alpha xy^3 + y\cos x) = \frac{\partial}{\partial x}(x^2y^2 + \beta\sin x)$

i.e., if $3\alpha xy^2 + \cos x = 2xy^2 + \beta\cos x$

$\therefore 3\alpha = 2$ & $\beta = 1$, equating coeff. of like terms.

$\therefore \alpha = \frac{2}{3}$ & $\beta = 1$

Note:- when $\alpha = \frac{2}{3}$, $\beta = 1$ given d. eqn becomes,

$$\left(\frac{2}{3}xy^3 + y\cos x\right)dx + (x^2y^2 + \sin x)dy = 0$$ which is exact.

$$\Rightarrow \frac{2}{3}xy^3dx + x^2y^2dy + y\cos x\,dx + \sin x\,dy = 0$$

$$\Rightarrow \frac{1}{3}\left(y^3.2x\,dx + x^2.3y^2dy\right) + y\cos x\,dx + \sin x.1.dy = 0$$

$$\Rightarrow \frac{1}{3}d(x^2y^3) + d(y\sin x) = 0$$

$$\therefore \frac{1}{3}\int d(x^2y^3) + \int d(y\sin x) = c$$

$$\Rightarrow \frac{1}{3}x^2y^3 + y\sin x = c$$ which is reqd. solution.

$$\Rightarrow \frac{1}{3}\left[\frac{\partial}{\partial x}(x^2y^3)dx + \frac{\partial}{\partial y}(x^2y^3)dy\right] + \frac{\partial}{\partial x}(y\sin x)dx + \frac{\partial}{\partial y}(y\sin x)dy = 0$$

$$\Rightarrow \frac{1}{3}\left[\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right] + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$ where $u = x^2y^3$, $v = y\sin x$

$$\Rightarrow \frac{1}{3}du + dv = 0$$

$$\Rightarrow \frac{1}{3}d(x^2y^3) + d(y\sin x) = 0$$

Integrating we get

$$\frac{1}{3}x^2y^3 + y\sin x = c$$, which is reqd. solution.