Exact differential equation of first order:-

A differential equation is said to be exact if it can be obtained from its primitive by direct differentiation only and without any further process of elimination or multiplication etc.

A d. eqn of first order of the form $Mdx + Ndy = 0$ is said to be an exact d. eqn if there exists a function $u(x,y)$ such that $Mdx + Ndy = du$ i.e., $Mdx + Ndy$ is a perfect or exact differential, where M & N are fns x & y.

e.g., $xdy + ydx = 0$ is an exact d. eqn because if $u = xy$ then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = ydx + xdy$.

Theorem:- The necessary and sufficient condition that the differential equation $Mdx + Ndy = 0$ to be exact is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ or $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$, where M & N are functions of x & y.

Proof:- Necessary part:

Let $Mdx + Ndy = 0$ —— (1) is an exact d. eqn where M and N are functions of x and y possessing continuous first order partial derivatives.

$\therefore$ there exists a function $u(x,y)$ such that

$$Mdx + Ndy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \text{—— (2)}$$

Since x and y are independent variables for the function $u(x,y)$ so dx & dy are also independent and therefore, equating coefficients of dx & dy from both sides of (2) we get $M = \frac{\partial u}{\partial x}$, $N = \frac{\partial u}{\partial y}$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial N}{\partial x}$$

i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ the condition is necessary.

Sufficient part:-

Let $Mdx + Ndy = 0$ —① is a d. eqn where M & N are functions of x & y possessing continuous first order partial derivatives such that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{—②}$$

To prove d. eqn ① is exact.

Let $v = \int_{y\text{-const.}} M\,dx$, then $\frac{\partial v}{\partial x} = M$.

From ②, $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

Integrating w.r.t. x we get,

$N = \frac{\partial v}{\partial y} + \varphi(y)$ where $\varphi(y)$ is a function of y alone and free from x.

$$\therefore Mdx + Ndy = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \varphi(y)dy$$

$$= dv + d\int \varphi(y)\,dy$$

$$= d\left[v + \int \varphi(y)\,dy\right]$$

$$= du \quad \text{where } u = v + \int \varphi(y)\,dy$$

i.e., $Mdx + Ndy$ is a perfect or exact differential.

$\therefore$ d. eqn ① is exact.

$\therefore$ the condition is sufficient.

Method of solution of $Mdx+Ndy=0$ when it is exact:-

First method:- When $Mdx+Ndy=0$ is exact d. eqn there exist a function $u(x,y)$ such that,

$$Mdx+Ndy = du$$
$$= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

So that $\frac{\partial u}{\partial x} = M$ —— (1) , $\frac{\partial u}{\partial y} = N$ —— (2)

Integrating (1) w.r.t. x we get-

$$u = \int_{y\text{-const}} Mdx + \phi(y) \quad \text{—— (3)}$$

Differentiating partially w.r.t. y both sides of (3) and using $\frac{\partial u}{\partial y} = N$ we can find $\phi(y)$ and then the solution of $Mdx+Ndy=0$ is given by $u(x,y)=c$.

2nd method (Inspection method):-

When $Mdx+Ndy=0$ is exact, by grouping the terms, the LHS of this d. eqn can be expressed as a perfect differential or as a sum of two or more perfect differentials and then integrating we get the required solution.

Note:-

(i) $ydx+xdy = d(xy)$

(ii) $xdx+ydy = \frac{1}{2}d(x^2+y^2)$

(iii) $\frac{xdy-ydx}{x^2} = d\left(\frac{y}{x}\right)$

(iv) $\frac{ydx-xdy}{y^2} = d\left(\frac{x}{y}\right)$

Justification (iv):- $\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{y\frac{dx}{dx} - x\frac{dy}{dx}}{y^2} = \frac{y - x\frac{dy}{dx}}{y^2}$

$$\therefore d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}.$$

Ex. 1. Show that the d. eq^n $(2x^3+4y)\,dx+(4x+y-1)\,dy=0$ is exact and hence solve it.

Soln. Given d. eq^n is $(2x^3+4y)\,dx+(4x+y-1)\,dy=0$ —— ①

or $M\,dx+N\,dy=0$. (say)

Here $M=2x^3+4y$, $N=4x+y-1$

$\therefore \dfrac{\partial M}{\partial y}=4$, $\dfrac{\partial N}{\partial x}=4$

$\therefore \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$.

$\therefore$ given d. eq^n is exact. *

Now, $(2x^3+4y)\,dx+(4x+y-1)\,dy=0$

$\Rightarrow 2x^3dx+4(y\,dx+x\,dy)+y\,dy-dy=0$

$\Rightarrow 2x^3dx+4\,d(xy)+y\,dy-dy=0$

$\therefore \int 2x^3dx+4\int d(xy)+\int y\,dy-\int dy=C$

$\Rightarrow 2\cdot\dfrac{x^4}{4}+4xy+\dfrac{y^2}{2}-y=C$

$\Rightarrow \dfrac{x^4}{2}+4xy+\dfrac{y^2}{2}-y=C$ which is required solution.

* **Alternative method:-**

Let $u=\int\limits_{y\text{-const.}} M\,dx=\int\limits_{y\text{-const.}}(2x^3+4y)\,dx$

$=2\cdot\dfrac{x^4}{4}+4yx+\phi(y)$ where $\phi(y)$ is a function of y-alone.

$=\dfrac{x^4}{2}+4xy+\phi(y)$

$\therefore \dfrac{\partial u}{\partial y}=4x+\phi'(y)$

$\Rightarrow N=4x+\phi'(y)$

$\Rightarrow 4x+y-1=4x+\phi'(y)$

$\Rightarrow \phi'(y)=y-1$

$\therefore \phi(y)=\int(y-1)\,dy=\dfrac{y^2}{2}-y$

[Note that when $M\,dx+N\,dy=0$ is exact then $M\,dx+N\,dy=du$ & $\dfrac{\partial u}{\partial y}=N,\ \dfrac{\partial u}{\partial x}=M$.]

$\therefore$ Solution is given by $u=C$.

or $\dfrac{x^4}{2}+4xy+\dfrac{y^2}{2}-y=C$.

Note:— $\phi'(y) = y-1$

$\therefore \phi(y) = \int (y-1)dy + c_1 = \frac{y^2}{2} - y + c_1$

$\therefore$ Solution is given by $u = c_2$

$\Rightarrow \frac{x^4}{2} + 4xy + \frac{y^2}{2} - y + c_1 = c_2$

$\Rightarrow \frac{x^4}{2} + 4xy + \frac{y^2}{2} - y = c$ where $c = c_2 - c_1$.

Ex. 2. Examine if the d. eqn $y\sin 2x\,dx - (1+y+\cos^2 x)dy = 0$ is exact and solve it.

Soln

$y\sin 2x\,dx - (1+y+\cos^2 x)dy = 0$ —— (1)

Here $M = y\sin 2x$, $N = -(1+y+\cos^2 x)$

$\therefore \frac{\partial M}{\partial y} = \sin 2x$, $\frac{\partial N}{\partial x} = -2\cos x(-\sin x) = \sin 2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ given d. eqn is exact.

Now, $y\sin 2x\,dx - (1+y+\cos^2 x)dy = 0$

$\Rightarrow 2y\sin x\cos x\,dx - \cos^2 x\,dy - dy - y\,dy = 0$

$\Rightarrow -(\cos^2 x\,dy - 2y\sin x\cos x\,dx) - dy - y\,dy = 0$

$\Rightarrow -\{\cos^2 x\,dy + y.2\cos x(-\sin x)dx\} - dy - y\,dy = 0$

$\Rightarrow -d(y\cos^2 x) - dy - y\,dy = 0$

$\Rightarrow d(y\cos^2 x) + dy + y\,dy = 0$

Integrating we get

$y\cos^2 x + y + \frac{y^2}{2} = C$.

which is required solution.

[Note that— $\frac{d}{dx}(y\cos^2 x) = \cos^2 x\frac{dy}{dx} + y.2\cos x(-\sin x)$

$\therefore d(y\cos^2 x) = \cos^2 x\,dy - y\sin 2x\,dx$.]

Alternative method:—

Let $u = \int_{y\text{-const.}} M\,dx = \int_{y\text{-const.}} y\sin 2x\,dx$

$= -\frac{y}{2}\cos 2x + \phi(y)$.

$\therefore \dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cos 2x + \phi'(y)$

$\Rightarrow N = -\dfrac{1}{2}\cdot(2\cos^2 x - 1) + \phi'(y)$

$\Rightarrow -(1+y+\cos^2 x) = -\cos^2 x + \dfrac{1}{2} + \phi'(y)$

$\Rightarrow \phi'(y) = -y - \dfrac{3}{2}$

$\therefore \phi(y) = -\int\left(y+\dfrac{3}{2}\right)dy = -\dfrac{y^2}{2} - \dfrac{3}{2}y$

$\therefore u = -\dfrac{1}{2}y\cos 2x - \dfrac{y^2}{2} - \dfrac{3}{2}y$

$\therefore$ Solution is given by $u = c$

or $-\dfrac{1}{2}y\cos 2x - \dfrac{y^2}{2} - \dfrac{3}{2}y = c$

or $y\cos 2x + y^2 + 3y + 2c = 0.$

2nd alternative:

Let $u = \int\limits_{x\text{-const}} N\,dy = \int\limits_{x\text{-const}} -(1+y+\cos^2 x)\,dy$

$= -y - \dfrac{y^2}{2} - y\cos^2 x + \phi(x)$

$\therefore \dfrac{\partial u}{\partial x} = -y\cdot 2\cos x(-\sin x) + \phi'(x)$

* $\Rightarrow M = y\sin 2x + \phi'(x)$

$\Rightarrow y\sin 2x = y\sin 2x + \phi'(x)$

$\Rightarrow \phi'(x) = 0$

$\therefore \phi(x) = \text{constant} = c_1$ (say)

$\therefore u = -y - \dfrac{y^2}{2} - y\cos^2 x + c_1$

$\therefore$ sol^n is given by $u = c_2$

or $-y - \dfrac{y^2}{2} - y\cos^2 x + c_1 = c_2$

or $-y - \dfrac{y^2}{2} - y\cos^2 x = c_2 - c_1 = c$ (say)

or $2y + y^2 + 2y\cos^2 x + 2c = 0$

or $2y + y^2 + y(1+\cos 2x) + 2c = 0$

or $y\cos 2x + y^2 + 3y + 2c = 0.$

* Note: that when $M\,dx + N\,dy = 0$ is exact then $M\,dx + N\,dy = du$ & $M = \dfrac{\partial u}{\partial x}$, $N = \dfrac{\partial u}{\partial y}$.

Ex.3. Show that the d.eqn

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$$x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$$ is exact and hence solve it.

Soln. $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2+y^2} = 0$

or $\left(x - \dfrac{y}{x^2+y^2}\right)dx + \left(y + \dfrac{x}{x^2+y^2}\right)dy = 0$ —— (1)

Here $M = x - \dfrac{y}{x^2+y^2}$, $N = y + \dfrac{x}{x^2+y^2}$

$\therefore \dfrac{\partial M}{\partial y} = 0 - \dfrac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}.$

$\dfrac{\partial N}{\partial x} = 0 + \dfrac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\therefore$ given d.eqn is exact.

Now, $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2+y^2} = 0$

$\Rightarrow x\,dx + y\,dy + \dfrac{\frac{x\,dy - y\,dx}{x^2}}{1+\frac{y^2}{x^2}} = 0$

$\Rightarrow x\,dx + y\,dy + \dfrac{d(y/x)}{1+(\frac{y}{x})^2} = 0$

$\therefore \int x\,dx + \int y\,dy + \int \dfrac{d(\frac{y}{x})}{1+(\frac{y}{x})^2} = C$

$\Rightarrow \dfrac{x^2}{2} + \dfrac{y^2}{2} + \tan^{-1}\dfrac{y}{x} = C$ which is reqd. soln.

Ex.4. Show that the d. eqn.

Home work: $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$ is exact & solve it.

Ans. $x + ye^{x/y} = C.$

Ex.5. Solve: $(xy\cos xy + \sin xy)\,dx + x^2\cos xy\,dy = 0$

Home work: Ans. $x\sin xy = C.$

Soln. of Ex.4. Here $M = 1 + e^{\frac{x}{y}}$, $N = e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)$

$\therefore \frac{\partial M}{\partial y} = e^{\frac{x}{y}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$, $\frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(0 - \frac{1}{y}\right) + \left(1 - \frac{x}{y}\right) \cdot e^{\frac{x}{y}} \cdot \frac{1}{y}$

$= e^{\frac{x}{y}} \cdot \left(-\frac{x}{y^2}\right)$ $\quad = \cancel{-\frac{1}{y}e^{\frac{x}{y}} + \frac{1}{y}e^{\frac{x}{y}}} - \frac{x}{y^2}e^{\frac{x}{y}}$

$= -\frac{x}{y^2}e^{\frac{x}{y}}$. $\quad = -\frac{x}{y^2}e^{\frac{x}{y}}$

Note, $\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So given d. eq^n is exact.

Let $u = \int_{y\text{-const.}} M\,dx = \int_{y\text{-const.}} \left(1 + e^{\frac{x}{y}}\right)dx = x + y.e^{\frac{x}{y}} + \phi(y)$

$\Rightarrow u = x + ye^{\frac{x}{y}} + \phi(y)$ —— (*)

$\therefore \frac{\partial u}{\partial y} = 0 + 1.e^{\frac{x}{y}} + y.e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right) + \phi'(y)$

$\Rightarrow N = e^{x/y} - \frac{x}{y}e^{\frac{x}{y}} + \phi'(y)$

$\Rightarrow \cancel{e^{x/y}\left(1 - \frac{x}{y}\right)} = \cancel{e^{x/y}\left(1 - \frac{x}{y}\right)} + \phi'(y)$

$\Rightarrow \phi'(y) = 0$

$\therefore \phi(y) = C_1$, constant.

$\therefore u = x + ye^{x/y} + \phi(y) = x + ye^{x/y} + C_1$

$\therefore$ Solution is given by $u = C_2$

$\Rightarrow x + ye^{x/y} + C_1 = C_2$

or $x + ye^{x/y} = C$ where $C = C_2 - C_1$ is arbitrary const.

Alt. $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - x/y\right)dy = 0$

$\Rightarrow \left(1 + y.e^{x/y} \cdot \frac{1}{y}\right)dx + \left\{1.e^{\frac{x}{y}} + ye^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)\right\}dy = 0$

$\Rightarrow \frac{\partial}{\partial x}\left(x + ye^{\frac{x}{y}}\right)dx + \frac{\partial}{\partial y}\left(x + ye^{\frac{x}{y}}\right)dy = 0$

$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$ where $u = x + ye^{x/y}$

$\Rightarrow du = 0 \quad \therefore u = C$ or $x + ye^{x/y} = C$ which is reqd soln.

Soln of Ex. 5. Here $M = xy\cos xy + \sin xy$, $N = x^2\cos xy$

$$\therefore \frac{\partial M}{\partial y} = x\cos xy + xy.(-\sin xy).x + \cos xy.x$$

$$= 2x\cos xy - x^2 y\sin xy$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2\cos xy) = 2x\cos xy + x^2(-\sin xy).y$$

$$= 2x\cos xy - x^2 y\sin xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$\therefore$ given d. eqn is exact.

Given d. eqn is, $(xy\cos xy + \sin xy)dx + x^2\cos xy\,dy = 0$

$$\Rightarrow x\cos xy(y\,dx + x\,dy) + \sin xy\,dx = 0$$

$$\Rightarrow x\cos xy\,d(xy) + \sin xy\,dx = 0$$

$$\Rightarrow \frac{\cos xy}{\sin xy}d(xy) + \frac{dx}{x} = 0$$

$$\therefore \int\frac{\cos xy}{\sin xy}d(xy) + \int\frac{dx}{x} = \log c$$

$$\Rightarrow \log(\sin xy) + \log x = \log c$$

$$\Rightarrow \log(x\sin xy) = \log c$$

$\therefore$ $x\sin xy = c$ which is reqd soln.

Alternative

Let $u = \int_{x\text{-const.}} N\,dy = \int_{x\text{-const.}} x^2\cos xy\,dy = x^2 \cdot \frac{1}{x}\sin xy + \phi(x)$

$$\Rightarrow u = x\sin xy + \phi(x)$$

$$\therefore \frac{\partial u}{\partial x} = 1.\sin xy + x.y\cos xy + \phi'(x)$$

$$\Rightarrow xy\cos xy + \sin xy = \sin xy + xy\cos xy + \phi'(x) \quad \because \frac{\partial u}{\partial x} = M$$

$\Rightarrow \phi'(x) = 0$, so $\phi(x) = c_1$, constant.

$$\therefore u = x\sin xy + c_1$$

$\therefore$ soln is $u = c_2$ or $x\sin xy + c_1 = c_2$

or $x\sin xy = c$ where $c = c_2 - c_1$

Ex. 6. Show that the d. eqⁿ $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$ is exact and hence solve it.
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Solⁿ.

Here $M = (e^y+1)\cos x$, $N = e^y \sin x$

$\therefore \frac{\partial M}{\partial y} = e^y \cos x$, $\frac{\partial N}{\partial x} = e^y \cos x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ given d. eqⁿ is exact.

Now we solve the d. eqⁿ:

$$(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$$

$$\Rightarrow (e^y+1)\cos x + e^y \sin x \frac{dy}{dx} = 0$$

$$\Rightarrow (e^y+1)\frac{d}{dx}\sin x + \sin x \cdot \frac{d}{dx}(e^y+1) = 0$$

$$\Rightarrow \frac{d}{dx}\{(e^y+1)\sin x\} = 0$$

Integrating w.r.t. x we get

$(e^y+1)\sin x = c$ which is reqd solⁿ.

Alternative :- $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

$$\Rightarrow \frac{\partial}{\partial x}\{(e^y+1)\sin x\}dx + \frac{\partial}{\partial y}\{(e^y+1)\sin x\}dy = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \quad \text{where } u = (e^y+1)\sin x$$

$$\Rightarrow du = 0$$

$$\therefore \int du = c$$

$$\Rightarrow u = c$$

$$\Rightarrow (e^y+1)\sin x = c, \text{ which is reqd. sol}^n.$$

2nd Alternative :- (general method)

Let $u = \int\limits_{y\text{-const.}} M\,dx = \int (e^y+1)\cos x\,dx$

$$\Rightarrow u = (e^y+1)\sin x + \phi(y)$$

$$\therefore \frac{\partial u}{\partial y} = e^y \sin x + \phi'(y)$$

$\Rightarrow N = e^y \sin x + \phi'(y)$, $\because \frac{\partial u}{\partial y} = N$

$\Rightarrow \cancel{e^y \sin x} = \cancel{e^y \sin x} + \phi'(y)$

$\Rightarrow \phi'(y) = 0$

$\therefore \phi(y) = C_1$, constant.

$\therefore u = (e^y + 1)\sin x + C_1$

$\therefore$ soln is given by $u = C_2$, constant

$\Rightarrow (e^y+1)\sin x + C_1 = C_2$

or $(e^y+1)\sin x = C$ where $C = C_2 - C_1$ is arbitrary constant.

Ex.7. '19H solve : $(x+y+1)dx + (x-y)dy = 0$, (2m)

Here $M = x+y+1$, $N = x-y$

$\therefore \frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so the d. eqn is exact.

Now, $(x+y+1)dx + (x-y)dy = 0$

$\Rightarrow x\,dx + dx - y\,dy + y\,dx + x\,dy = 0$

$\Rightarrow x\,dx + dx - y\,dy + d(xy) = 0$ *

$\therefore \int x\,dx + \int dx - \int y\,dy + \int d(xy) = C$

$\Rightarrow \frac{x^2}{2} + x - \frac{y^2}{2} + xy = C$

$\Rightarrow x^2 + 2x - y^2 + 2xy = 2C$ which is reqd. soln.

Ex.8. Solve: (i) $(ax+hy+g)dx + (hx+by+f)dy = 0$

2008 (ii) $(2x+2y-5)dy + (3x+2y-5)dx = 0$

Soln (ii) Here $M = 3x+2y-5$, $N = 2x+2y-5$ Note that coeff. of dx is M

$\therefore \frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 2$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so given d.eqn is exact.

given d. eqn is $(2x+2y-5)dy+(3x+2y-5)dx=0$

$$\Rightarrow 2(xdy+ydx)+(2y-5)dy+(3x-5)dx=0$$

$$\Rightarrow 2d(xy)+(2y-5)dy+(3x-5)dx=0$$

$$\therefore \int 2d(xy)+\int(2y-5)dy+\int(3x-5)dx=C$$

$$\Rightarrow 2xy+2\cdot\frac{y^2}{2}-5y+3\cdot\frac{x^2}{2}-5x=C$$

or $4xy+2y^2-10y+3x^2-10x=2C$

which is reqd. soln.

(i) Ans:- $ax^2+2hxy+by^2+2gx+2fy=2C$. (HW).

Ex. 9. solve: $(\cos y+y\cos x)dx+(\sin x-x\sin y)dy=0$
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Here $M=\cos y+y\cos x$, $N=\sin x-x\sin y$

$$\therefore \frac{\partial M}{\partial y}=-\sin y+\cos x, \quad \frac{\partial N}{\partial x}=\cos x-\sin y$$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, so the given d. eqn is exact.

Let $u=\int\limits_{y\text{-const.}} M\,dx=\int\limits_{y\text{-const}}(\cos y+y\cos x)dx$

$$\Rightarrow u=x\cos y+y\sin x+\phi(y)$$

$$\therefore \frac{\partial u}{\partial y}=-x\sin y+\sin x+\phi'(y)$$

$$\Rightarrow \cancel{\sin x}-\cancel{x\sin y}=-\cancel{x\sin y}+\cancel{\sin x}+\phi'(y) \quad ;\because \frac{\partial u}{\partial y}=N$$

$$\Rightarrow \phi'(y)=0$$

$\therefore \phi(y)=C_1$, constant

$$\therefore u=x\cos y+y\sin x+C_1$$

$\therefore$ soln is given by, $u=C_2$

or $x\cos y+y\sin x+C_1=C_2$

or $x\cos y+y\sin x=C_2-C_1=C$ (say)

or $x\cos y+y\sin x=C$

Ex. 10. Solve:
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$$\left(\frac{x}{x^2+y^2}+x^2y\right)dy+\left(xy^2-\frac{y}{x^2+y^2}\right)dx=0$$

Here, $M = xy^2 - \dfrac{y}{x^2+y^2}$, $N = \dfrac{x}{x^2+y^2} + x^2y$ [Note that coeff of dx is M & that of dy is N.]

$$\therefore \frac{\partial M}{\partial y} = 2xy - \frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2}$$

$$= 2xy - \frac{x^2-y^2}{(x^2+y^2)^2} = 2xy + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} + x^2y\right)$$

$$= \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} + 2xy = 2xy + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, so given d. eq^n is exact.

Now we solve the d. eq^n:

$$\left(\frac{x}{x^2+y^2}+x^2y\right)dy+\left(xy^2-\frac{y}{x^2+y^2}\right)dx=0$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2+y^2} + xy(x\,dy + y\,dx) = 0$$

$$\Rightarrow \frac{\frac{x\,dy-y\,dx}{x^2}}{1+\frac{y^2}{x^2}} + xy\,d(xy) = 0 \quad , \because \frac{x\,dy-y\,dx}{x^2} = d\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} + xy\,d(xy) = 0$$

$$\therefore \int \frac{du}{1+u^2} + \int v\,dv = c \qquad \text{where } u = \frac{y}{x},\ v = xy$$

$$\Rightarrow \tan^{-1}u + \frac{v^2}{2} = c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{1}{2}(xy)^2 = c$$

$$\Rightarrow 2\tan^{-1}\frac{y}{x} + x^2y^2 = 2c$$

which is reqd. solution.

Ex. 11. Solve : $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$

Soln. Here $M=y^2e^{xy^2}+4x^3$, $N=2xye^{xy^2}-3y^2$

$$\therefore \frac{\partial M}{\partial y}=2ye^{xy^2}+y^2.e^{xy^2}.x.2y+0$$
$$=2ye^{xy^2}(1+xy^2).$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(2xye^{xy^2}-3y^2)$$
$$=2ye^{xy^2}+2xy.e^{xy^2}.y^2-0$$
$$=2ye^{xy^2}(1+xy^2)$$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ & hence given d. eq^n is exact.

Let $u=\underset{y\text{-const.}}{\int} M\,dx=\underset{y\text{-const.}}{\int}(y^2e^{xy^2}+4x^3)dx$

$$=y^2.\frac{1}{y^2}e^{xy^2}+4.\frac{x^4}{4}+\varphi(y)$$
$$\Rightarrow u=e^{xy^2}+x^4+\varphi(y) \quad \text{———} \; (*)$$
$$\therefore \frac{\partial u}{\partial y}=e^{xy^2}.2xy+0+\varphi'(y)$$
$$\Rightarrow 2xye^{xy^2}-3y^2=2xye^{xy^2}+\varphi'(y), \because \frac{\partial u}{\partial y}=N.$$
$$\Rightarrow \varphi'(y)=-3y^2$$
$$\therefore \varphi(y)=\int -3y^2dy+c_1=-y^3+c_1$$
$$\therefore u=e^{xy^2}+x^4-y^3+c_1$$

$\therefore$ Solution is given by $u=c_2$

$$\Rightarrow e^{xy^2}+x^4-y^3+c_1=c_2$$

or $e^{xy^2}+x^4-y^3=c$ where $c=c_2-c_1$ is arbitrary constant.

Alternative.

$(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$

$\Rightarrow y^2e^{xy^2}dx+2xye^{xy^2}dy+4x^3dx-3y^2dy=0$

$\Rightarrow d(e^{xy^2})+4x^3dx-3y^2dy=0$

Integrating we get $e^{xy^2}+x^4-y^3=c$ which is reqd. solution.

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Note:- $y^2 e^{xy^2}dx + 2xy\, e^{xy^2}dy + 4x^3dx - 3y^2dy = 0$

$$\Rightarrow e^{xy^2}(y^2dx + 2xy\,dy) + 4x^3dx - 3y^2dy = 0$$

$$\Rightarrow e^{xy^2}d(xy^2) + 4x^3dx - 3y^2dy = 0$$

or $e^t dt + 4x^3dx - 3y^2dy = 0$ where $t = xy^2$

Integrating we get -

$$e^t + 4\cdot\frac{x^4}{4} - 3\cdot\frac{y^3}{3} = c$$

'12P. $\Rightarrow e^{xy^2} + x^4 - y^3 = c$ which is the soln.

Ex.12. Solve: $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$

Soln. Here $M = x^4 - 2xy^2 + y^4$, $N = -2x^2y + 4xy^3 - \sin y$

$\therefore \frac{\partial M}{\partial y} = 0 - 4xy + 4y^3$, $\frac{\partial N}{\partial x} = -4xy + 4y^3 - 0$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ given d.eqn is exact. and it is

$$(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$$

$$\Rightarrow x^4dx - \sin y\,dy - 2xy(y\,dx + x\,dy) + y^4dx + 4xy^3dy = 0$$

$$\Rightarrow x^4dx - \sin y\,dy - 2xy\,d(xy) + d(xy^4) = 0$$

Integrating we get -

$$\frac{x^5}{5} + \cos y - 2\frac{(xy)^2}{2} + xy^4 = c$$

$\Rightarrow \frac{x^5}{5} + \cos y - x^2y^2 + xy^4 = c$ which is reqd. solution.

Ex.13. Solve: $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$

Home work. Ans:— $\frac{x^4}{4} + \frac{y^4}{4} + \frac{3}{2}x^2y^2 = c.$

Ex.14. Solve: $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Home work: Ans:— $\frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c.$

19H **Ex.15.** Solve: $(x+y)^2dx + (x^2 + 2xy - y^2)dy = 0$

Home work: Ans:— $x^3 + 3x^2y + 3xy^2 - y^3 = 3c.$