Ex. 3. Form the differential equation by eliminating arbitrary constants a & b from $y = e^{-x}(a\cos x + b\sin x)$.

Soln.

$$y = e^{-x}(a\cos x + b\sin x) \text{ --- } (1)$$

$$\therefore \frac{dy}{dx} = -e^{-x}(a\cos x + b\sin x) + e^{-x}(-a\sin x + b\cos x)$$

$$\Rightarrow \frac{dy}{dx} = -y + e^{-x}(-a\sin x + b\cos x) \text{ --- } (2), \text{ using } (1)$$

Diff. w.r.t. x

$$\frac{d^2y}{dx^2} = -\frac{dy}{dx} + e^{-x}(-a\cos x - b\sin x) - e^{-x}(-a\sin x + b\cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{dy}{dx} - y - \left(\frac{dy}{dx} + y\right) \text{ using } (1) \text{ \& using } (2)$$

$$\Rightarrow \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$ which is required d. eq^n.

(This d.eq^n is of 2nd order & 1st degree)

Ex. 4. Form the differential equation whose primitive (solution) is (where a, b, c are arbitrary constants):–

(i) $$y = ae^{3x} + be^{-x} \text{ --- } (1)$$

$$\therefore \frac{dy}{dx} = 3ae^{3x} - be^{-x}$$

$$= 3(y - be^{-x}) - be^{-x} \text{ using } (1)$$

$$\Rightarrow \frac{dy}{dx} = 3y - 4be^{-x} \text{ --- } (2)$$

Differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = 3\frac{dy}{dx} + 4be^{-x} = 3\frac{dy}{dx} + \left(3y - \frac{dy}{dx}\right) \text{ using } (2)$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0,$$ which is reqd. d. eq^n.

(ii) $$(x-c)^2 + y^2 = 16 \text{ --- } (1)$$

Diff. w.r.t. x

$$2(x-c) + 2y\frac{dy}{dx} = 0$$

$\Rightarrow x - c = -y\frac{dy}{dx}$ —— (2)

From (1) & (2),

$$\left(-y\frac{dy}{dx}\right)^2 + y^2 = 16$$

or $y^2\left\{\left(\frac{dy}{dx}\right)^2 + 1\right\} = 16.$ which is reqd. d. eqn.

(This d. eqn is of 1st order & 2nd degree)

(iii) $y = a\cos(\log x) + b\sin(\log x)$ —— (1)   in this primitive there are two arbitrary constants.

$\therefore \frac{dy}{dx} = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$

$\Rightarrow x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$

Diff. w.r.t. x

$x\frac{d^2y}{dx^2} + 1\cdot\frac{dy}{dx} = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\{a\cos(\log x) + b\sin(\log x)\}$

$= -y$ using (1)

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ which is reqd. d. eqn.

[Note that in forming a d. eqn we can differentiate the primitive as many times as the number of independent arbitrary constants.]

(iv) $y^2 - 2ay + x^2 = a^2$ —— (1) , in this primitive there is only one arbitrary const.

Diff. w.r.t. x

$2y\frac{dy}{dx} - 2a\frac{dy}{dx} + 2x = 0$

$\Rightarrow yy_1 - ay_1 + x = 0$ where $y_1 = \frac{dy}{dx}$

$\Rightarrow ay_1 = x + yy_1$

$\Rightarrow a = \frac{1}{y_1}(x + yy_1)$ —— (2)

From ① & ②,

$$y^2 - \frac{2}{y_1}(x+yy_1)y + x^2 = \frac{1}{y_1^2}(x+yy_1)^2$$

$$\Rightarrow y^2y_1^2 - 2(x+yy_1)yy_1 + x^2y_1^2 = (x+yy_1)^2$$

$$\Rightarrow \cancel{y^2y_1^2} - 2xyy_1 - 2y^2y_1^2 + x^2y_1^2 - x^2 - 2xyy_1 - \cancel{y^2y_1^2} = 0$$

$$\Rightarrow (x^2-2y^2)y_1^2 - 4xyy_1 - x^2 = 0$$

or $(x^2-2y^2)\left(\frac{dy}{dx}\right)^2 - 4xy\frac{dy}{dx} - x^2 = 0$. which is reqd. d. eqn.

(v) $x = a\cos nt + b\sin nt$ — ① Here only a & b are arbitrary constants.

$$\therefore \frac{dx}{dt} = -na\sin nt + nb\cos nt$$

$$\therefore \frac{d^2x}{dt^2} = -n^2a\cos nt - n^2b\sin nt$$

$$= -n^2(a\cos nt + b\sin nt) = -n^2x \quad \text{using ①}$$

$$\Rightarrow \frac{d^2x}{dt^2} + n^2x = 0 \quad \text{which is reqd. d. eqn.}$$

(vi) $y = a\cos x + b\sin x + x\sin x$ (HW)

Ans. $\frac{d^2y}{dx^2} + y = 2\cos x$.

(vii) $y^2 = 4a(x+a)$ — ① , 'a' is the only arbitrary constant.

Diff. w.r.t. x

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow a = \frac{1}{2}y\frac{dy}{dx} \quad \text{— ②}$$

From ① & ②, we get—

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{1}{2}y\frac{dy}{dx}\right)$$

$$\Rightarrow y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

or $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ which is reqd. d. eqn.

(viii) $y = a\cos(mx+b)$ —— ① , only a & b are arbitrary constants.

$\therefore \frac{dy}{dx} = -ma\sin(mx+b)$

$\therefore \frac{d^2y}{dx^2} = -m^2 a\cos(mx+b) = -m^2y$ using ①

$\Rightarrow \frac{d^2y}{dx^2} + m^2y = 0$ which is reqd. d. eq^n.

(ix) $xy = ae^x + be^{-x} + x^2$ —— ①

Diff. w.r.t. x

$x\frac{dy}{dx} + 1.y = ae^x - be^{-x} + 2x$

Diff. w.r.t. x

$x\frac{d^2y}{dx^2} + 1.\frac{dy}{dx} + \frac{dy}{dx} = ae^x + be^{-x} + 2$

$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2$ using ①

$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 2 - x^2$ which is reqd. d. eq^n.

(x) $y = (a+bx)e^{3x}$ —— ①

$\therefore \frac{dy}{dx} = 3(a+bx)e^{3x} + be^{3x}$

$\Rightarrow \frac{dy}{dx} = 3y + be^{3x}$ —— ② using ①

Diff. w.r.t. x

$\frac{d^2y}{dx^2} = 3\frac{dy}{dx} + 3be^{3x}$

$= 3\frac{dy}{dx} + 3\left(\frac{dy}{dx} - 3y\right)$ using ②

$\Rightarrow \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ which is reqd. d. eq^n.

(xi) 2019 (Jm) $y = mx + c$ —— ① where m is fixed c is a parameter (i.e., only c is arbitrary const.)

$\therefore \frac{dy}{dx} = m$ which is required. d. eq^n.

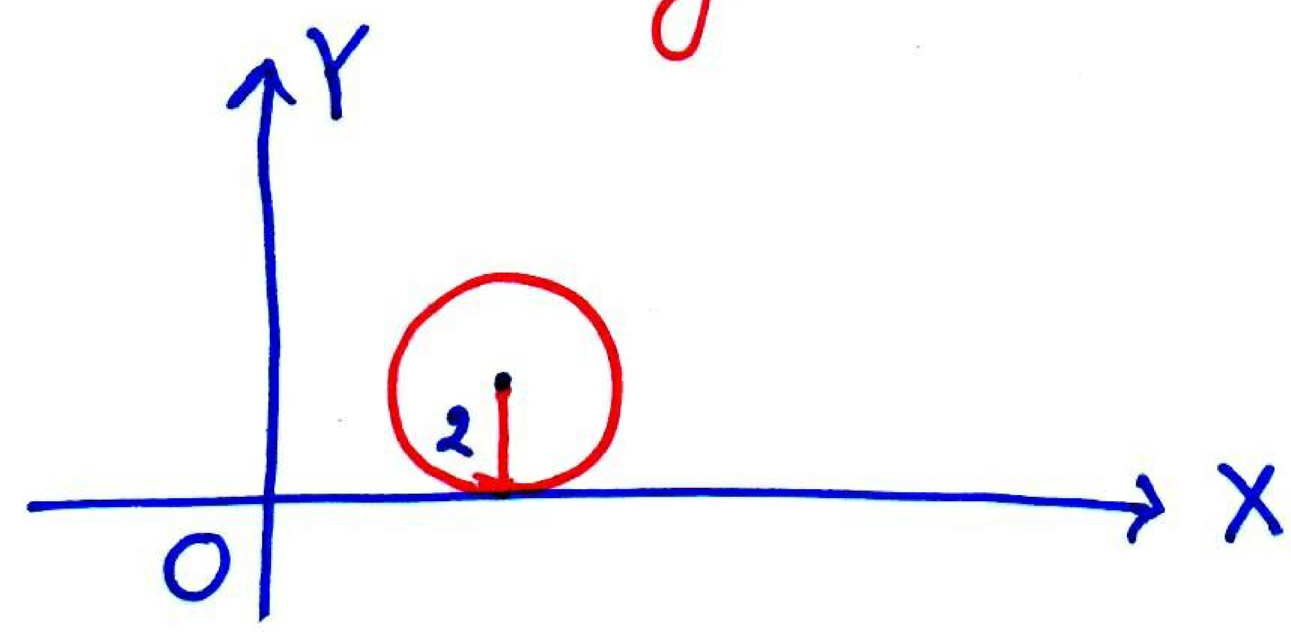
Ex. 5. What is the differential equation of the family of all straight lines parallel to y-axis.

Soln. Equation of the family of all straight lines parallel to y-axis is given by $x = c$ where c a parameter i.e., arbitrary const.

$$\therefore \quad \frac{dx}{dy} = 0$$

which is required d. eqn.

Ex. 6. Find the differential equation of the family of all circles in the first quadrant touching x-axis and having radius 2.



Soln. Equation of the family of given circles is :

$$(x-h)^2 + (y-2)^2 = 2^2 \quad \text{— (1)}$$ where h is a parameter.

Diff. w.r.t. x

$$2(x-h) + 2(y-2)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x - h = -(y-2)\frac{dy}{dx} \quad \text{— (2)}$$

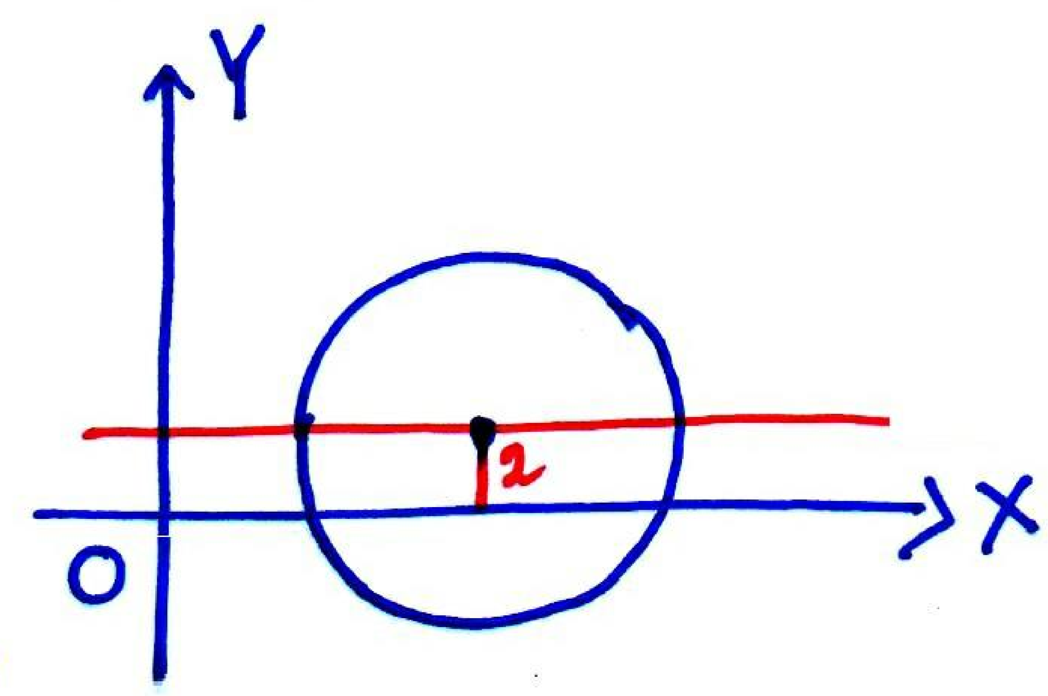
From (1) & (2) we get,

$$(y-2)^2\left(\frac{dy}{dx}\right)^2 + (y-2)^2 = 4$$

or $\quad (y-2)^2\left\{\left(\frac{dy}{dx}\right)^2 + 1\right\} = 4$ which is reqd. d. eqn.

Ex. 7. Find the differential equation of the family of all circles with radius 5 and centres on the line $y = 2$.

Soln. Equation of the family of given circles is



$$(x-h)^2 + (y-2)^2 = 5^2 \quad \text{— (1)}$$

which is similar to (1) of ex. 6.

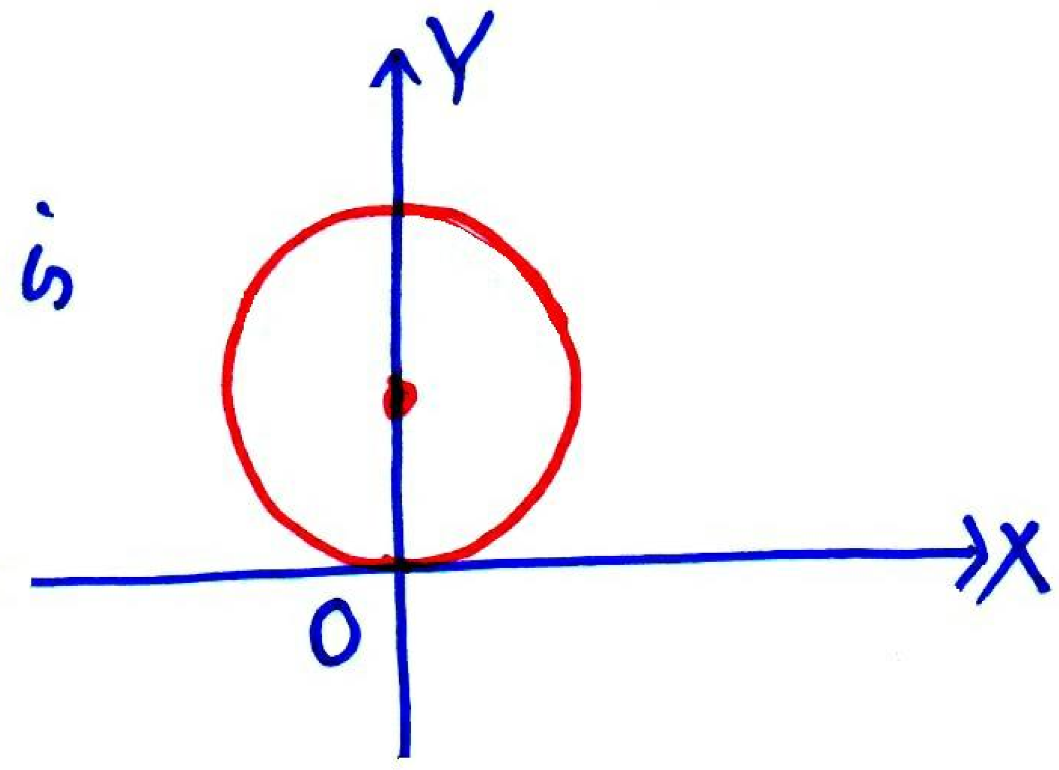
Proceeding as in ex. 6. reqd. d. eqn is

$$(y-2)^2\left\{\left(\frac{dy}{dx}\right)^2 + 1\right\} = 25.$$

Ex. 8. Find the differential equation of the family of all circles passing through the origin and having centres on y-axis.

OR

Find the differential equation of the family of all circles touching x-axis at origin.



Soln. Equation of the family of given circles is

$$x^2+(y-k)^2=k^2 \quad \text{——} (1)$$

where k is a parameter.

Diff. (1) w.r.t. x

$$2x+2(y-k)\frac{dy}{dx}=0$$

$$\Rightarrow \quad y-k=-\frac{x}{y_1} \quad \text{——} (2) \quad \text{where } y_1=\frac{dy}{dx}$$

$$\Rightarrow \quad k=y+\frac{x}{y_1} \quad \text{——} (3)$$

Using the values from (2) & (3) in (1) we get

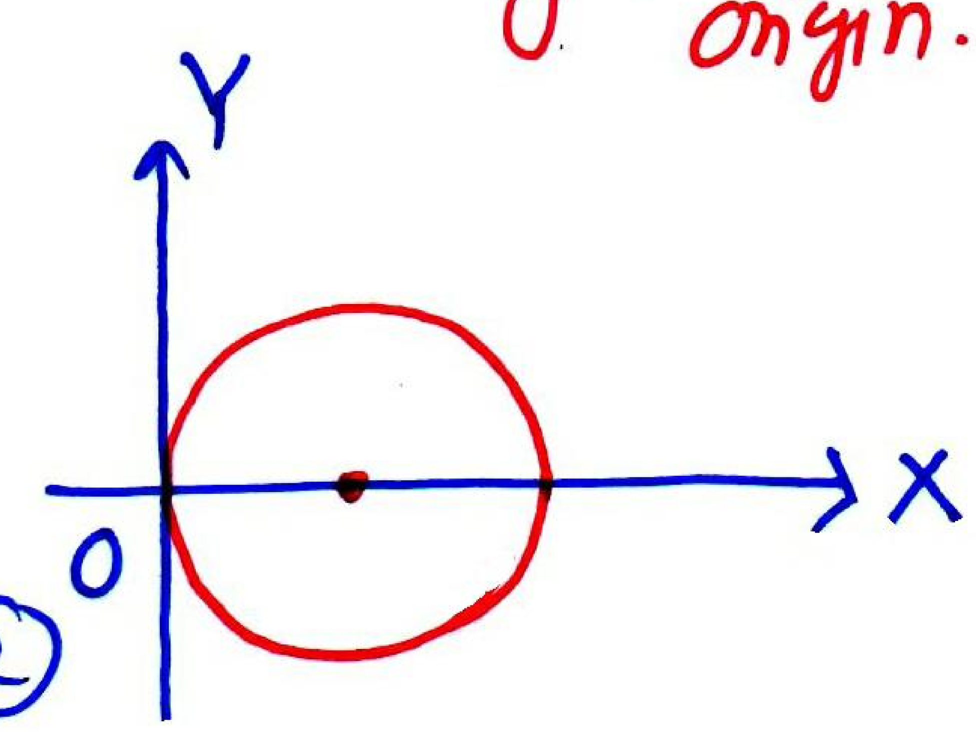
$$x^2+\cancel{\frac{x^2}{y_1^2}}=\left(y+\frac{x}{y_1}\right)^2=y^2+2y.\frac{x}{y_1}+\cancel{\frac{x^2}{y_1^2}}$$

$$\Rightarrow \quad x^2-y^2=2xy.\frac{1}{\frac{dy}{dx}}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2xy}{x^2-y^2} \quad \text{which is reqd. d. eqn.}$$

Ex. 9. Find the differential equation of the family of all circles passing through the origin and having centres on x-axis

OR Find the d. eqn of the family of all circles touching y-axis at origin.



Soln. Equation of the family of given circles is

$$(x-h)^2+y^2=h^2 \quad \text{——} (1)$$ where h is a parameter.

Diff. w.r.t. x

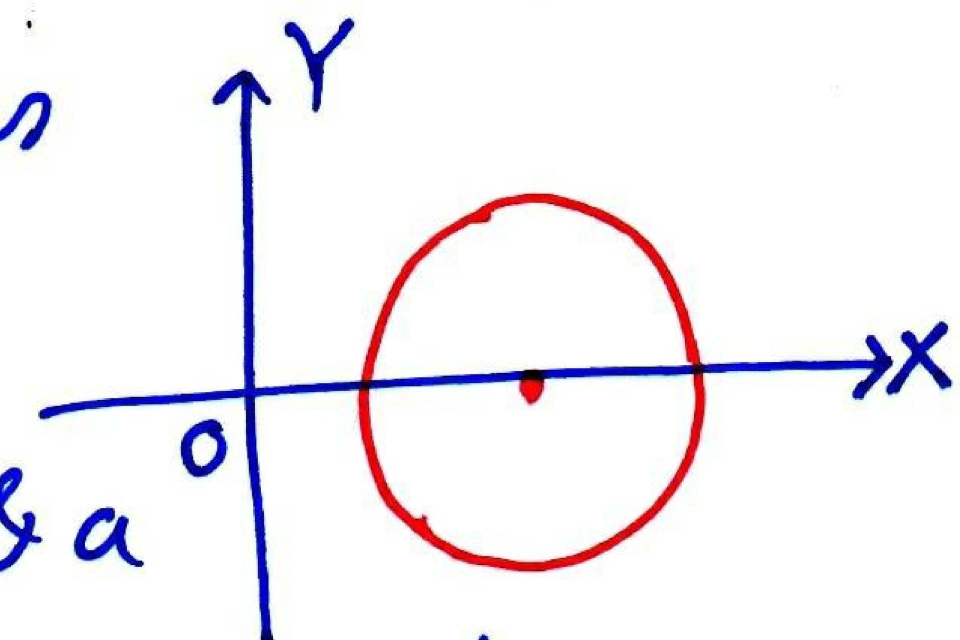
$$2(x-h)+2yy_1=0 \Rightarrow x-h=-yy_1 \quad \text{——} (2)$$

From (1) & (2), $y^2y_1^2+y^2=(x+yy_1)^2 \Rightarrow y^2=x^2+2xyy_1$

or $2xy\frac{dy}{dx}+x^2=y^2$ which is reqd. d. eqn.

Ex. 10. Find the differential equation of the family of all circles (i) having centres on x-axis (ii) having centres on y-axis.

Soln (i) Equation of the family of all circles with centres on x-axis is given by



$$(x-h)^2+y^2=a^2 \quad \text{—— (1)}$$ where h & a are both parameters.

Diff. w.r.t. x

$$2(x-h)+2y\frac{dy}{dx}=0$$

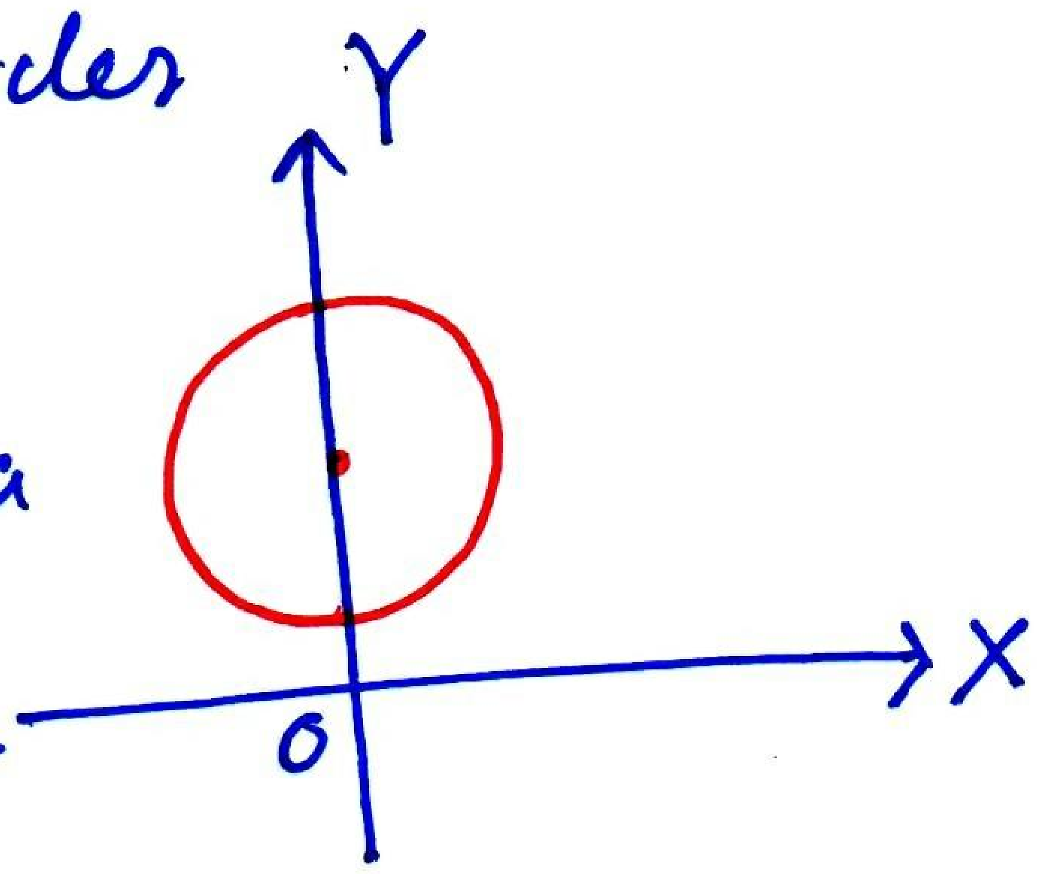
$$\Rightarrow \quad x-h+y\frac{dy}{dx}=0$$

Diff. w.r.t. x

$$1+\left(y\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx}\right)=0$$

$$\Rightarrow y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0$$ which is reqd. d. eqn.

(ii) Equation of the family of all circles with centres on y-axis is given by



$$x^2+(y-k)^2=a^2 \quad \text{—— (1)}$$ where k & a are both parameters.

Diff. w.r.t. x

$$2x+2(y-k)\frac{dy}{dx}=0$$

$$\Rightarrow \quad x+(y-k)\frac{dy}{dx}=0 \quad \text{—— (2)}$$

Diff. w.r.t. x

$$1+(y-k)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow 1-\frac{x}{\frac{dy}{dx}}\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0 \quad \text{using (2)}$$

$$\Rightarrow \quad \frac{dy}{dx}-x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$$ which is reqd. d. eqn.

Ex. 11. Find the order and degree of the differential equation representing the family of curves $y^2=2c(x+\sqrt{c})$.

soln. $y^2=2c(x+\sqrt{c})$ —— (1) where c is a parameter.

Diff. w.r.t. x

$$2y\frac{dy}{dx} = 2c \Rightarrow y\frac{dy}{dx} = c \quad \text{---} \ (2)$$

From (1) & (2) we get,

$$y^2 = 2y\frac{dy}{dx}\left(x + \sqrt{y\frac{dy}{dx}}\right)$$

$$\Rightarrow y - 2x\frac{dy}{dx} = 2\frac{dy}{dx}\sqrt{y\frac{dy}{dx}}$$

$$\Rightarrow \left(y - 2x\frac{dy}{dx}\right)^2 = 4\left(\frac{dy}{dx}\right)^2 \cdot y\frac{dy}{dx}$$

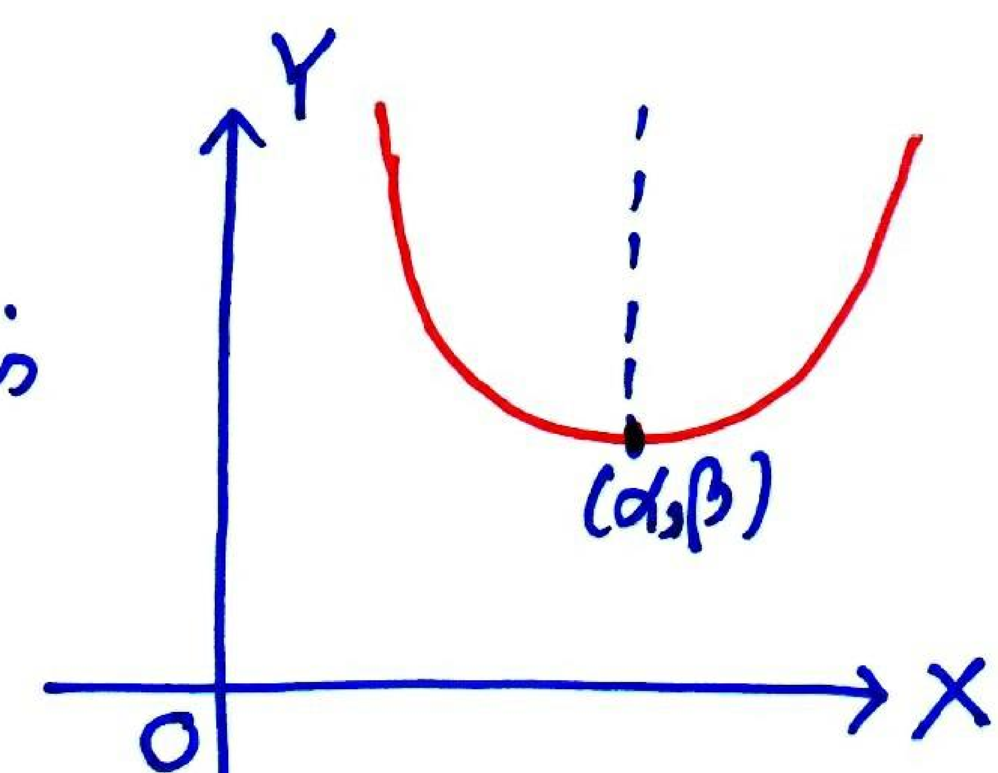
$$\Rightarrow y^2 - 4xy\frac{dy}{dx} + 4x^2\left(\frac{dy}{dx}\right)^2 = 4y\left(\frac{dy}{dx}\right)^3$$

$$\Rightarrow 4y\left(\frac{dy}{dx}\right)^3 - 4x^2\left(\frac{dy}{dx}\right)^2 + 4xy\frac{dy}{dx} - y^2 = 0$$ which is reqd. d. eqn.

This d. eqn is of 1st order & 3rd degree.

Ex. 12. Find the differential equation of the family of all parabolas whose axes are parallel to (i) x-axis (ii) y-axis. Also state order & degree of this d. eqn.

Soln. (i) The equation of the family of parabolas with axis parallel to y-axis is given by $(x-\alpha)^2 = 4a(y-\beta)$ ---- (1)

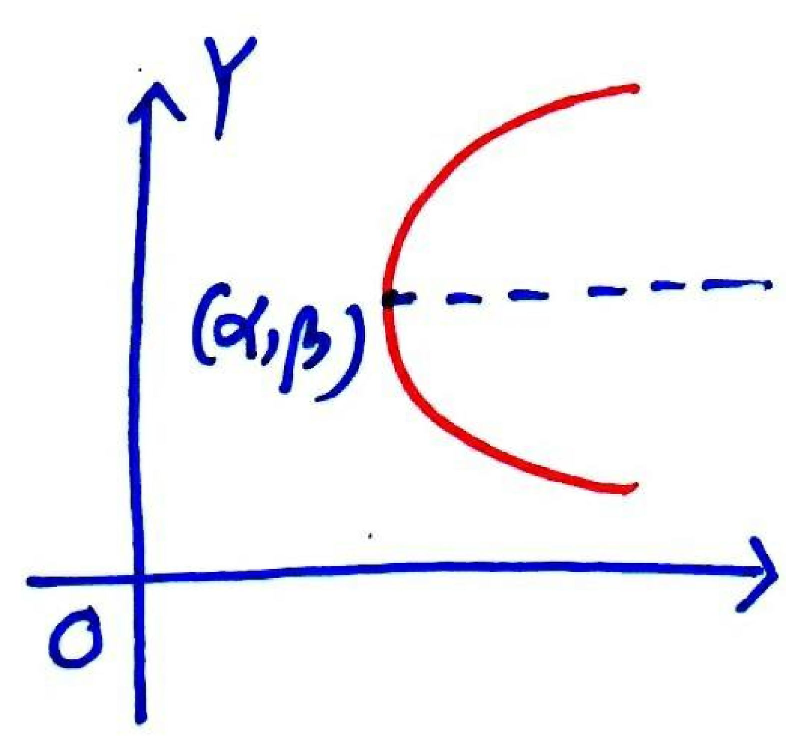


where α, β, a are three parameters.

Diff. w.r.t. x

$$2(x-\alpha) = 4a\frac{dy}{dx}$$

$$\Rightarrow x - \alpha = 2a\frac{dy}{dx} \quad \text{---} \ (2)$$



Diff. w.r.t. x

$$1 = 2a\frac{d^2y}{dx^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2a} \quad \text{---} \ (3)$$

Diff. w.r.t. x

$$\frac{d^3y}{dx^3} = 0$$ which is reqd. d. eqn and its order is 3 & degree is 1.

(ii) Eqn of the family of parabolas is $(y-\beta)^2 = 4a(x-\alpha)$. etc. (HW)

Ans:- $y_1 y_3 - 3y_2^2 = 0$, order 3 degree 1.

By BN Sir

Ex. 13. Find the differential equation of the family of curves (confocal conics) $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ where λ is parameter.

Soln.

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \quad \text{—— (1)}$$

$$\Rightarrow (b^2+\lambda)x^2+(a^2+\lambda)y^2=(a^2+\lambda)(b^2+\lambda)$$

Diff. w.r.t. x

$$(b^2+\lambda)\cdot 2x+(a^2+\lambda)2yy_1=0 \quad \text{where } y_1=\frac{dy}{dx}$$

$$\Rightarrow \lambda(x+yy_1)=-(b^2x+a^2yy_1)$$

$$\Rightarrow \lambda=-\frac{b^2x+a^2yy_1}{x+yy_1} \quad \text{—— (2)}$$

$$\therefore a^2+\lambda = a^2-\frac{b^2x+a^2yy_1}{x+yy_1}=\frac{a^2x+\cancel{a^2yy_1}-b^2x-\cancel{a^2yy_1}}{x+yy_1}$$

$$\Rightarrow a^2+\lambda=\frac{(a^2-b^2)x}{x+yy_1} \quad \text{—— (3)}$$

Also $$b^2+\lambda=b^2-\frac{b^2x+a^2yy_1}{x+yy_1}=\frac{\cancel{b^2x}+b^2yy_1-\cancel{b^2x}-a^2yy_1}{x+yy_1}$$

$$\Rightarrow b^2+\lambda=\frac{-(a^2-b^2)yy_1}{x+yy_1} \quad \text{—— (4)}$$

From (1), (3) & (4) we get —

$$\frac{x^2(x+yy_1)}{(a^2-b^2)x}-\frac{y^2(x+yy_1)}{(a^2-b^2)yy_1}=1$$

$$\Rightarrow (x+yy_1)\left(x-\frac{y}{y_1}\right)=a^2-b^2$$

$$\Rightarrow \left(x+y\frac{dy}{dx}\right)\left(x-\frac{y}{\frac{dy}{dx}}\right)=a^2-b^2$$

$$\Rightarrow \left(x+y\frac{dy}{dx}\right)\left(x\frac{dy}{dx}-y\right)=(a^2-b^2)\frac{dy}{dx}$$

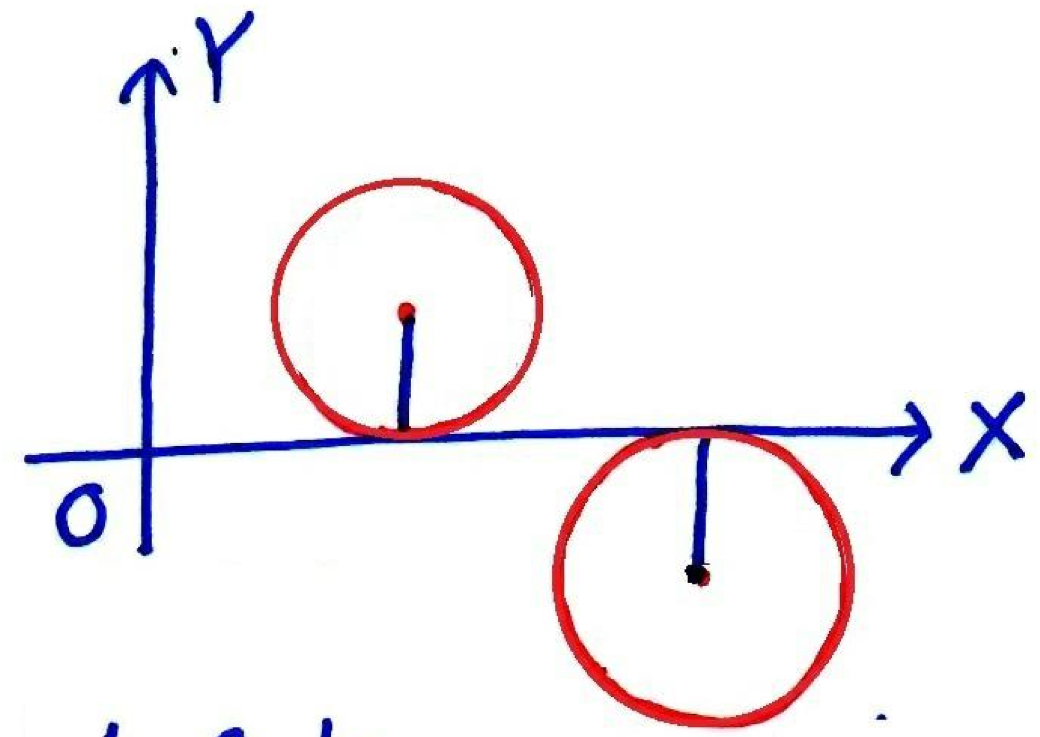
$$\Rightarrow xy\left(\frac{dy}{dx}\right)^2+(x^2-y^2+b^2-a^2)\frac{dy}{dx}-xy=0$$ which is reqd. d.eqn.

[The order and degree of this d.eqn are 1 & 2 respectively]

By BN Sir

Ex. 14. (2019) Find the differential equation of the family of all circles touching the x-axis. (4 marks)

Soln. Equation of the family of circles touching x-axis is given by



$$(x-h)^2+(y-k)^2=k^2 \quad \text{—— (1)}$$

where h & k are parameters i.e., arbitrary constants & radius is $|k|$.

Diff. w.r.t. x

$$2(x-h)+2(y-k)\frac{dy}{dx}=0$$

$$\Rightarrow \; x-h+(y-k)y_1=0 \quad \text{—— (2)} \qquad \text{where } y_1=\frac{dy}{dx}$$

Diff. w.r.t. x

$$1+(y-k)y_2+y_1\cdot y_1=0 \qquad \text{where } y_2=\frac{dy_1}{dx}=\frac{d^2y}{dx^2}$$

$$\Rightarrow \; y-k=-\frac{1}{y_2}(1+y_1^2) \quad \text{—— (3)}$$

From (1), (2) & (3) we get,

$$(y-k)^2y_1^2+(y-k)^2=k^2 \quad , \because x-h=-(y-k)y_1$$

$$\Rightarrow \; (y_1^2+1)(y-k)^2=k^2$$

$$\Rightarrow \; (y_1^2+1)\frac{1}{y_2^2}(1+y_1^2)^2=\left\{y+\frac{1}{y_2}(1+y_1^2)\right\}^2$$

$$=y^2+2\frac{y}{y_2}(1+y_1^2)+\frac{1}{y_2^2}(1+y_1^2)^2$$

$$\Rightarrow \; (1+y_1^2)^3=y^2y_2^2+2yy_2(1+y_1^2)+(1+y_1^2)^2$$

$$\Rightarrow \; \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3-\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2-2y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d^2y}{dx^2}-y^2\left(\frac{d^2y}{dx^2}\right)^2=0$$

which is reqd. d. eqn.

Ex. 15. Obtain the differential equation whose solution is
2019

$$y = a\cos x + b\sin x + \frac{1}{x}(b\cos x - a\sin x).$$

Soln.

$$y = a\cos x + b\sin x + \frac{1}{x}(b\cos x - a\sin x)$$

or $$xy = x(a\cos x + b\sin x) + b\cos x - a\sin x \quad \text{--- (1)}$$

Diff. w.r.t. x both sides of (1) we get,

$$x\frac{dy}{dx} + 1\cdot y = \cancel{a\cos x} + \cancel{b\sin x} + x(-a\sin x + b\cos x) - \cancel{b\sin x} - \cancel{a\cos x}$$

$$\Rightarrow xy_1 + y = x(-a\sin x + b\cos x) \quad \text{--- (2)}$$

Diff. w.r.t. x

$$xy_2 + 1\cdot y_1 + y_1 = x(-a\cos x - b\sin x) - a\sin x + b\cos x$$

$$\Rightarrow xy_2 + 2y_1 = (-xy + b\cos x - a\sin x) - a\sin x + b\cos x \quad \text{using (1)}$$

$$\Rightarrow xy_2 + 2y_1 + xy = 2(-a\sin x + b\cos x) \quad \text{--- (3)}$$

$$\Rightarrow x^2y_2 + 2xy_1 + x^2y = 2x(-a\sin x + b\cos x)$$

$$= 2(xy_1 + y) \quad \text{using (2)}$$

$$\Rightarrow x^2y_2 + x^2y - 2y = 0$$

or $$x^2\frac{d^2y}{dx^2} + (x^2 - 2)y = 0 \quad \text{which is reqd. d. eqn.}$$

Ex. 16. (i) 2022 Find the differential equation whose solution (primitive) is the family of curves defined by $x^2+y^2+2ax+2by+c=0$ where a, b, c are parameters. (3m)

soln.

$$x^2+y^2+2ax+2by+c=0 \quad \text{——} \ (1)$$

Diff. w. r. t. x

$$2x+2yy_1+2a+2by_1=0 \quad \text{where } y_1=\frac{dy}{dx}$$

$$\Rightarrow \quad x+yy_1+a+by_1=0$$

$$\Rightarrow \quad x+(y+b)y_1+a=0 \quad \text{——} \ (2)$$

Diff. w. r. t. x

$$1+y_1.y_1+(y+b)y_2=0 \quad \text{where } y_2=\frac{d^2y}{dx^2}=\frac{dy_1}{dx}$$

$$\Rightarrow \quad 1+y_1^2+(y+b)y_2=0 \quad \text{——} \ (3)$$

Diff. w. r. t. x

$$2y_1y_2+y_1y_2+(y+b)y_3=0 \quad \text{where } y_3=\frac{dy_2}{dx}=\frac{d^3y}{dx^3}$$

$$\Rightarrow \quad 3y_1y_2+(y+b)y_3=0 \quad \text{——} \ (4)$$

$$(3)\times y_3-(4)\times y_2 \Rightarrow (1+y_1^2)y_3-3y_1y_2^2=0$$

$$\Rightarrow \left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2=0$$

which is required d. eqn.

Ex. 16. (ii) 2022 Form the differential equation for the curve $y=k\sin^{-1}x$, k being a constant. (arbitrary constant/ parameter) (2m)

soln

$$y=k\sin^{-1}x \quad \text{——} \ (1)$$

$$\therefore \frac{dy}{dx}=k.\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}} \quad \text{using } (1)$$

$$\Rightarrow \sqrt{1-x^2}\sin^{-1}x\frac{dy}{dx}=y$$

$$\Rightarrow \sqrt{1-x^2}\sin^{-1}x\frac{dy}{dx}-y=0 \quad \text{which is reqd. d. eqn.}$$