Differential Equation:- *An equation involving differentials or derivatives of one or more dependent variables w.r.t. one or more independent variables, is called a* ***differential equation***.

A differential equation involving ordinary derivatives of one or more dependent variables w.r.t. a single independent variable is called an ***Ordinary differential equation***.

A differential equation involving partial derivatives of one or more dependent variables w.r.t. more than one independent variables is called a ***Partial differential equation***.

e.g. $(i)\quad \frac{dy}{dx}+2y=sinx\ ;\quad (ii)\quad \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3-xy=e^x\quad (iii)\quad xdy+ydx=0,$

are **ordinary** differential equations.

$(iv)\quad \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=3z\ ;\quad (v)\quad \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}-2\frac{\partial^2 z}{\partial x\partial y}=0\quad (vi)\quad \frac{\partial^2 u}{\partial t^2}=k\left(\frac{\partial^2 u}{\partial x^2}\right)^3$, k is a constant.

are **partial** differential equations.

Order of a Differential Equation:-*The order of a differential equation is the order of the highest derivative or differential present in the equation.*

e.g., the differential equation (i) , (iii) & (iv) above are 1st order and (ii), (v) & (vi) are of 2nd order.

Degree of an Algebraic Differential Equation:- *The degree of an algebraic differential equation (in which derivatives or differentials involved appear as polynomials)is defined as the degree of the highest order derivative or differential present in the equation after the equation is made* ***free*** *from radicals and fractions as far as the derivatives are concerned.*

Linear & Non-linear Ordinary Differential Equation:- *A differential equation is called linear if (i) every dependent variable & every derivative involved occur to the first degree only, (ii) no products of dependent variables or derivatives occur, (iii) no transcendental functions of dependent variables or their derivatives occur.(* ***transcendental functions*** *are functions other than algebraic functions, e.g., trigonometric , logarithmic, exponential, hyperbolic functions etc.)*

A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x, is an equation of the form :

$P_0(x)\frac{d^ny}{dx^n}+P_1(x)\frac{d^{n-1}y}{dx^{n-1}}+P_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\ \dots\dots\dots\ +P_{n-1}(x)\frac{dy}{dx}+P_n(x)y=Q(x)$ *where* $P_0(x)$

is not identically zero and $P_0(x),\ P_1(x),\ P_2(x),\dots\dots\dots,P_n(x),\ Q(x)$ *are functions of* x *only.*

A differential equation which is not linear is called a non-linear differential equation.

e.g., the differential equation (i), (iii), (iv) & (v) above are all linear d. equations & (ii) , (vi) are non-linear d. equations.

Ex.1. ***Classify each of the following differential equations as ordinary or partial, linear or non-linear differential equations, also state the order and degree of each of them:***

$(i)\quad \frac{d^2y}{dx^2}=1+\left(\frac{dy}{dx}\right)^{\frac{1}{3}}$. Ans: *It is an ordinary non-linear d. eqn of 2nd order & 3rd degree.*

Because it can be written as : $\frac{d^2y}{dx^2}-1=\left(\frac{dy}{dx}\right)^{\frac{1}{3}}$ *or,* $\left(\frac{d^2y}{dx^2}-1\right)^3=\frac{dy}{dx}$.

$(ii)\quad xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0.$ Ans: *It is an ordinary non-linear d. eqn of 2nd order & 1st degree.*

$(iii)\quad \frac{\partial^2 z}{\partial x^2}=\sqrt[3]{z+\left(\frac{\partial z}{\partial y}\right)^3}$. Ans: *It is a non-linear Partial d.eqn of 2nd order & 3rd degree.*

Because it can be written as : $\left(\frac{\partial^2 z}{\partial x^2}\right)^3=z+\left(\frac{\partial z}{\partial y}\right)^3$

(iv) $\frac{dy}{dx}+\frac{\sqrt{\frac{d^2y}{dx^2}+y}}{\frac{dy}{dx}}+y=0.$ Ans: *It is an ordinary non-linear d. eq^n of 2nd order & 1st degree.*

Because it can be written as : $\left(\frac{dy}{dx}\right)^2+y\frac{dy}{dx}=-\sqrt{\frac{d^2y}{dx^2}+y}$ *or* $\left\{\left(\frac{dy}{dx}\right)^2+y\frac{dy}{dx}\right\}^2=\frac{d^2y}{dx^2}+y\,.$

(v) $\frac{\partial^3 z}{\partial x^3}+k\frac{\partial^2 z}{\partial y^2}=0\,.$ Ans: *It is a linear Partial d.eq^n of 3nd order & 1st degree.*

(vi) $\left(\frac{\partial^3 z}{\partial x^3}\frac{\partial z}{\partial y}\right)^2=1+\left(\frac{\partial^2 z}{\partial x^2}\right)^2.$ Ans: *It is a non-linear Partial d.eq^n of 3rd order & 2nd degree.*

'19H (vii) $\frac{d^3y}{dx^3}+x^2\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=2\,.$ Ans: *It is an ordinary non-linear d. eq^n of 3rd order & 1st degree.*

(viii) $log\left(\frac{d^2y}{dx^2}\right)+cos\left(\frac{dy}{dx}\right)^3+y=x^2\,.$ Ans: *It is an ordinary non-linear d. eq^n of 2nd order whose degree cannot be ascertained.*

(ix) $log\left(\frac{d^2y}{dx^2}\right)+sinx=0.$ Ans: *It is an ordinary linear d. eq^n of 2nd order & 1st degree.*

Because it can be written as : $log\left(\frac{d^2y}{dx^2}\right)=-sinx,$ *or* $\frac{d^2y}{dx^2}=e^{-sinx}$

(x) $\left[1-\left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{3}{2}}=\frac{\partial^2 z}{\partial x^2}\,.$ Ans: *It is a non-linear Partial d.eq^n of 2nd order & 2nd degree.*

Because it can be written as(squaring both sides): $\left[1-\left(\frac{\partial z}{\partial y}\right)^2\right]^3=\left(\frac{\partial^2 z}{\partial x^2}\right)^2$

'22H (xi) $\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{3}}=\left(y+\frac{dy}{dx}\right)^{\frac{1}{2}}$. Ans: *It is an ordinary non-linear d. eq^n of 2nd order & 2nd degree. Because it can be written as(raising both sides to 6th power):* $\left(\frac{d^2y}{dx^2}\right)^2=\left(y+\frac{dy}{dx}\right)^3$

'22H (xii) $\sqrt{x}\,\frac{dy}{dx}+\frac{K}{\frac{dy}{dx}}=y$ Ans: *It is an ordinary non-linear d. eq^n of 1st order & 2nd degree.*

Because it can be written as : $\sqrt{x}\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}+K=0$

Formation of Ordinary Differential Equation:- Let $f(x,\ y,\ C)=0$ ……… (1) is a relation involving independent variable x, dependent variable y and arbitrary constant C. Differentiating (1) w.r.t. x we get a relation of the form

$\phi\left(x,\ y,\ \frac{dy}{dx},\ C\right)=0$ ……… (2) , eliminating C from (1) & (2) we get a relation of the form $\Psi\left(x,\ y,\ \frac{dy}{dx}\right)=0$ ………… (3) which is a differential equation of 1st order and the relation (1) is called its ***primitive or Solution.***

If $f(x,\ y,\ C_1,C_2)=0$ is a relation involving independent variable x, dependent variable y and independent arbitrary constants C_1 & C_2 then differentiating this relation w.r.t. x twice successively we get two more equations and eliminating C_1 & C_2 from the three equations we get a differential equation of 2nd order.

Similarly higher order d. equations can be formed from a given relation involving three or more arbitrary constants.

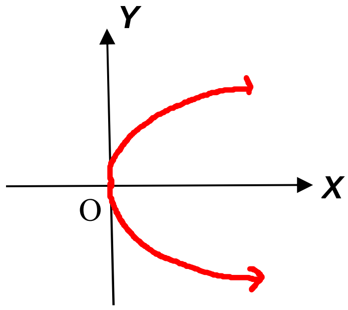
NOTE :- (i) The order of a differential equation is equal to the number of independent arbitrary constants in its primitive.

In forming an ordinary differential equation from its primitive we are to differentiate it as many times as the number of independent arbitrary constants (i.e., no. of parameters)

(ii) A differential equation in two variables geometrically represents a family of curves all satisfying some common properties.

Ex.2. Form the differential equation of the family of parabolas having the vertex at origin and axis along (i) the positive x-axis. (ii) the negative y-axis.
(iii) What is the order of the D.equation of a three-parameter family of curves? (2022)
(iv) Obtain the differential equation whose solution is $y = mx + c$ where m is fixed & c is a parameter(2022).

Sol.:-(i) The equation of the family of parabolas having the vertex at origin and axis along the positive x-axis is given by $y^2 = 4ax \ldots\ldots (i)$ where a is an arbitrary constant.



Differentiating (i) w.r.t. x we get

$$2y\frac{dy}{dx} = 4a$$

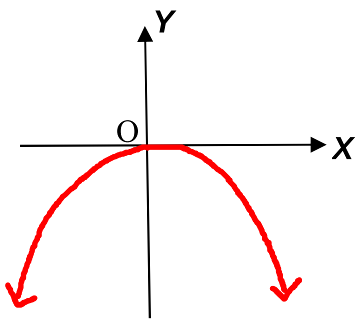
$$\Rightarrow 2xy\frac{dy}{dx} = 4ax$$

$$\Rightarrow 2xy\frac{dy}{dx} = y^2 \quad \text{using (i)}$$

$$\Rightarrow 2x\frac{dy}{dx} - y = 0$$ which is required d. equation.

Sol.:-(ii) The equation of the family of parabolas having the vertex at origin and axis along the negative y-axis is given by $x^2 = -4ay \ldots\ldots (i)$

Differentiating (i) w.r.t. x we get



$$2x = -4a\frac{dy}{dx}$$

$$\Rightarrow 2xy = -4ay\frac{dy}{dx}$$

$$\Rightarrow 2xy = x^2\frac{dy}{dx} \quad \text{using (i)}$$

$$\Rightarrow x\frac{dy}{dx} - 2y = 0$$ which is required d. equation.

Sol.:-(iii) Order is three because there are three parameters.

Sol.:-(iv) $y = mx + c,$

$diff.w.r.t.x$

$\frac{dy}{dx} = m$ which is required d. equation.